Dirac-mode Expansion for Confinement and Chiral Symmetry Breaking

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References:

S. Gongyo, T. Iritani, H.S.,
"Gauge-Invariant Formalism with Dirac-mode Expansion for Confinement and Chiral Symmetry Breaking", hep-lat:1202.4130.

H. S., S. Gongyo, T. Iritani, A. Yamamoto,
"Relevant Gluonic Momentum for Confinement and Gauge-Invariant Formalism with Dirac-mode Expansion", PoS (QCD-TNT11) 044 (2011).

Abstract: With the Dirac-mode expansion, we analyze the correspondence between confinement and chiral symmetry breaking in SU(3) lattice QCD. Notably, the confinement force is almost unchanged even after removing the low-lying Dirac modes, which are responsible to chiral symmetry breaking. This indicates that one-to-one correspondence does not hold for between confinement and chiral symmetry breaking in QCD.

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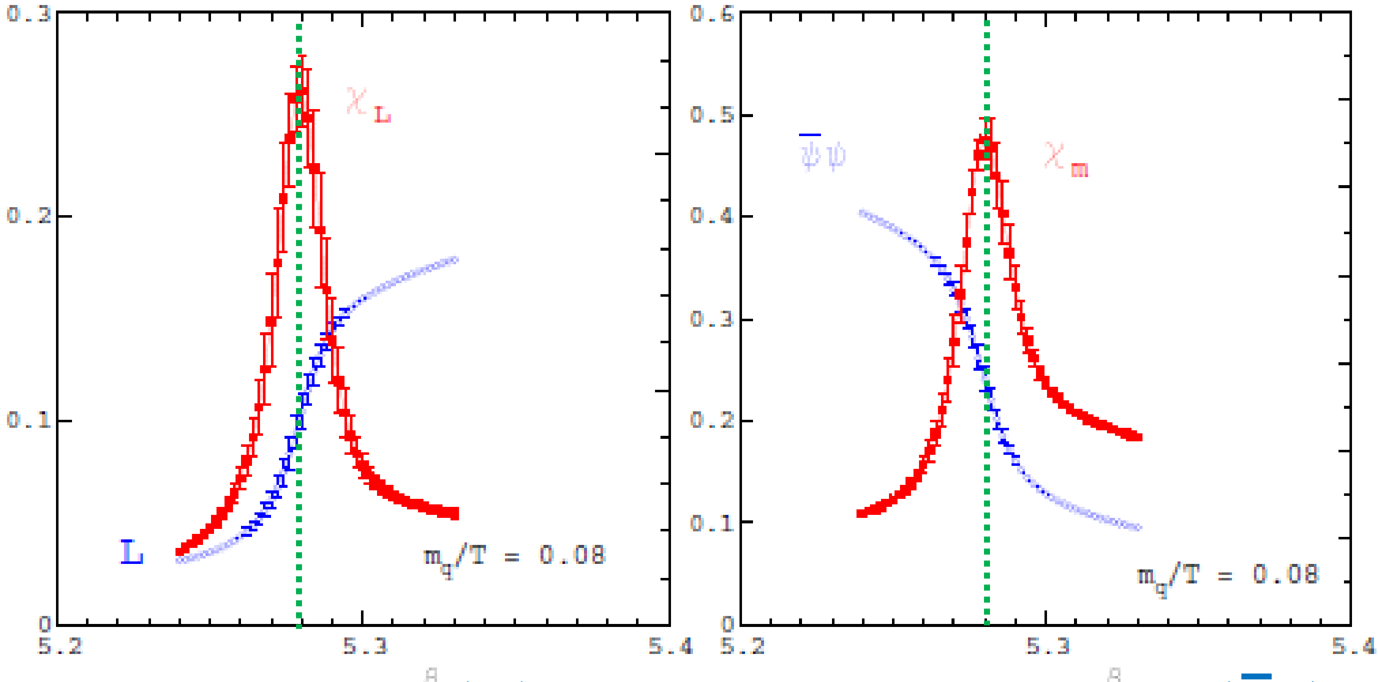
Introduction : Confinement and Chiral Symmetry Breaking

Color Confinement and Chiral Symmetry Breaking (CSB)
are Two of most important phenomena of
Nonperturbative QCD

The relation between
Confinement and CSB is not yet known
directly from QCD.

Correlation between Confinement and CSB is suggested by
Simultaneous Phase Transition of
Deconfinement and Chiral Restoration.

Lattice QCD results at finite temperature F. Karsch, Lect. Notes Phys. (2002)



Polyakov Loop <**P**> Chiral Condensate <**$\bar{q}$q**>

Color Confinement Chiral Symmetry Breaking

Fig. 2. Deconfinement and chiral symmetry restoration in 2-flavour QCD: Shown is $\langle L \rangle$ (left), which is the order parameter for deconfinement in the pure gauge limit ($m_q \to \infty$), and $\langle \bar{\psi}\psi \rangle$ (right), which is the order parameter for chiral symmetry breaking in the chiral limit ($m_q \to 0$). Also shown are the corresponding susceptibilities as a function of the coupling $\beta = 6/g^2$.

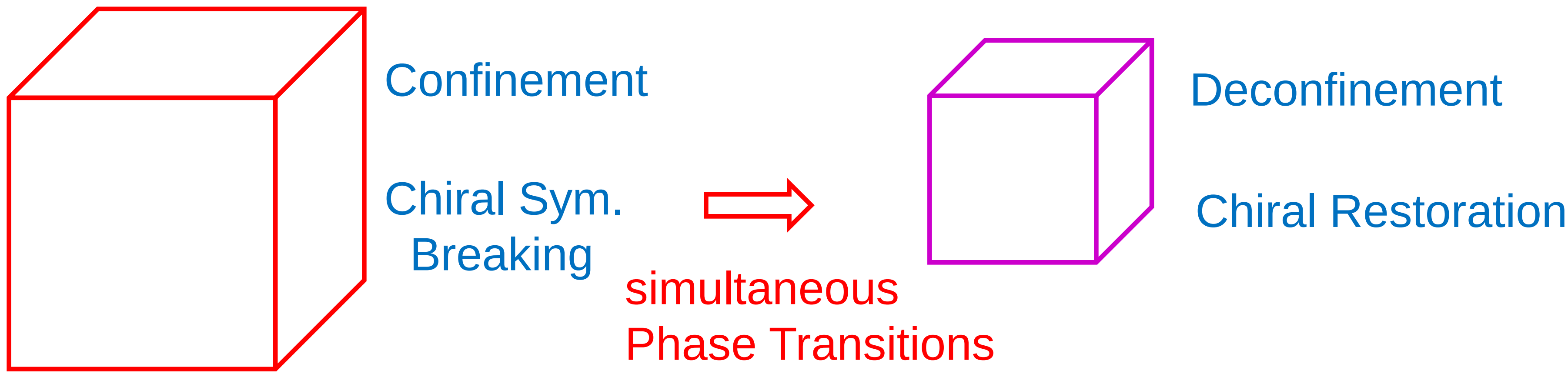
More on correlation between Confinement and Chiral Sym Breaking

Also, similar Coincidence between Deconfinement and Chiral Restoration is found in Finite-Size lattice QCD.

In fact, Simultaneous Phase Transitions occur according to the Box Size.

Large Volume Lattice

Small Volume Lattice



Of course, Finite-Temperature Phase transition is also a kind of Finite-Size effect of Euclidean Lattice in temporal direction.

More on correlation between Confinement and Chiral Sym Breaking

The close relation between Confinement and CSB has been indicated in terms of Monopoles appearing in Maximally Abelian Gauge in QCD.

By removing the Monopoles from the QCD vacuum, the confinement property and chiral symmetry breaking are simultaneously lost.

[e.g. Dual GL theory: H.S. et al, NPB (1995),
LQCD : O.Miyamura, PLB (1995), R.Woloshyn, PRD(1995),]

O. Miyamura

Important role of Monopole to Chiral Sym Breaking (Lattice QCD)

O. Miyamura, PLB (1995) :
First Lattice QCD Study to reveal Important role of Monopoles to CSB

Quark Condensate plotted against β in SU(2) QCD on16^3x 4 lattice

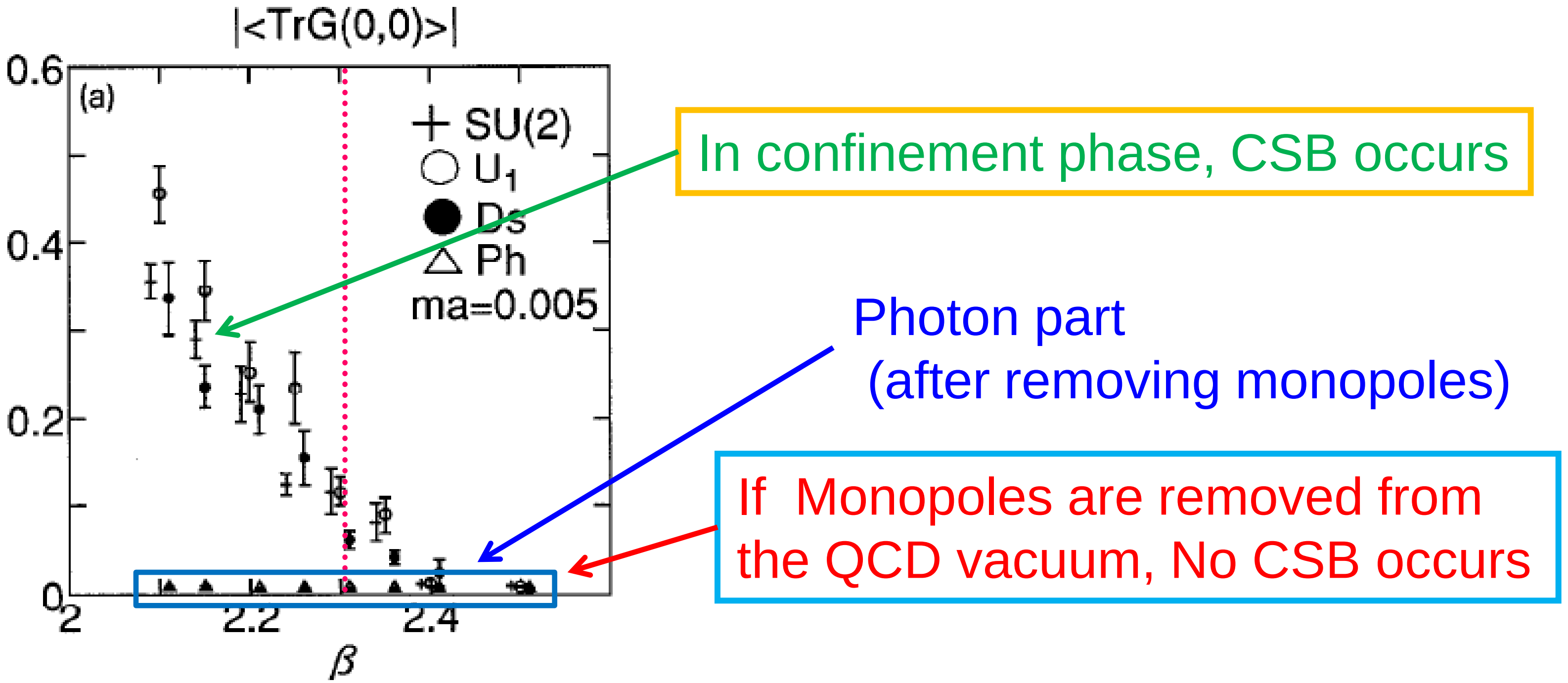
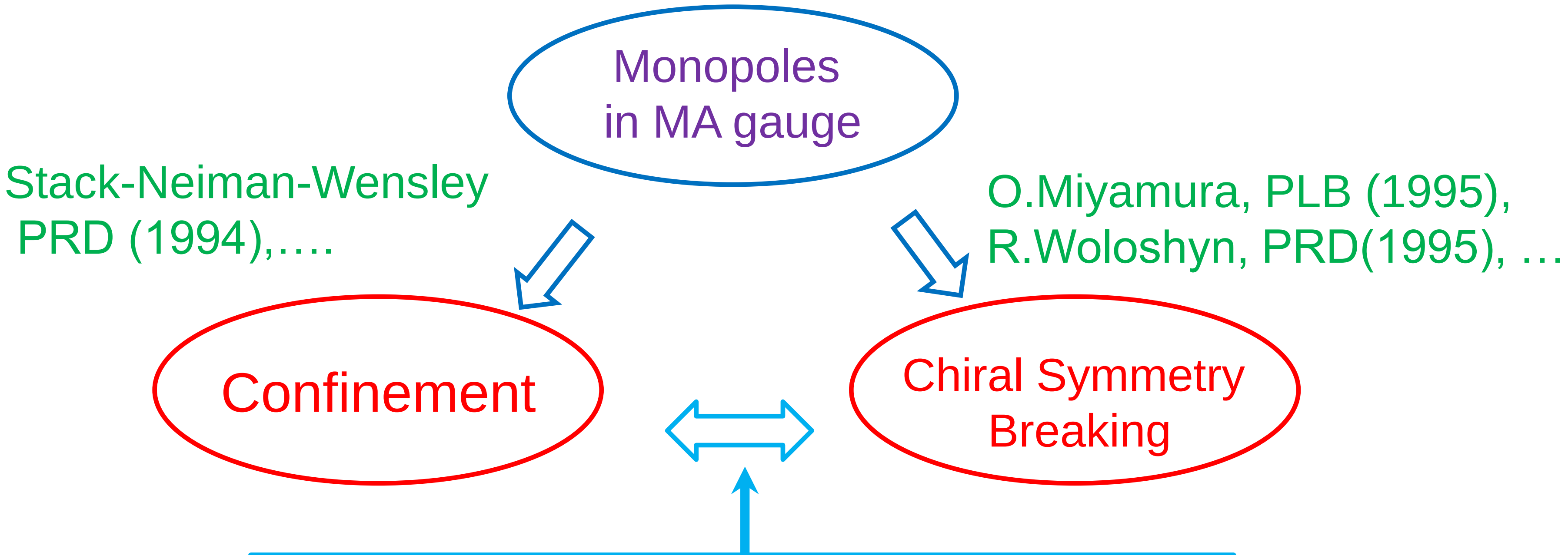


Fig. 4. (a) $|\langle \mathrm{Tr}\, G(0,0)\rangle|$ for $ma = 0.005$ in the SU(2) field (cross), in the U(1) field (open circle), its singular (filled circle) and regular (triangle) components on a $16^3 \times 4$ lattice. (b) Same for $ma = 0.01$.

Monopole part (including only monopole) : **Chiral sym breaking**
Photon part (after removing monopole) : **Chial Symmetric**

Relation between Confinement and Chiral Symmetry Breaking

The lattice QCD studies indicate an important role of monopoles to both Confinement and CSB, and these two nonperturbative phenomena seem to be related through the monopole.



We would like to know the relation between Confinement and CSB in more direct manner.

So, we investigate Confinement using the Dirac-mode expansion, because the essential modes for CSB are Low-lying Dirac modes.

Banks-Casher Relation

$$\Sigma \equiv \left|\langle \bar{q}q \rangle\right| = \lim_{m\to 0}\lim_{V\to\infty} \pi\,\rho(0)$$

$$\rho(\lambda) = \frac{1}{V}\left\langle \sum_k \delta(\lambda - \lambda_k)\right\rangle$$ **: QCD Dirac operator eigenvalue density**

Zero-eigenvalue density ρ(0) of Dirac operator
gives Chiral Condensate.
⇒ The essential modes for Chiral Sym Breaking are
Low-lying Dirac modes.

※ The non-zero spectrum is symmetric due to $\{\gamma_5, \not{D}\} = 0$

$$\because \not{D}\psi_n = \lambda_n \psi_n \rightarrow \not{D}(\gamma_5 \psi_n) = -\lambda_n (\gamma_5 \psi_n)$$

Eigen-mode of Dirac operator in Lattice QCD

$$\not{D}^{\rm lat}_{\rm xy} = \frac{1}{2a}\sum_{\mu=1}^{4}\gamma^{\mu}[U_{\mu}(x)\delta_{y,x+\hat{\mu}} - U_{-\mu}(x)\delta_{y,x-\hat{\mu}}]$$:Lattice Dirac operator

$$\not{D}^{\rm lat}[U]|n\rangle = \lambda_n|n\rangle$$:Dirac eigen-value, Dirac eigen-state

$$\sum_y \not{D}^{\rm lat}_{xy}[U]\psi_n(y) = \lambda_n\psi_n(x)$$:Dirac eigen-function ψ_n(x)

Explicit form of eigen-value equation in lattice QCD

$$\frac{1}{2a}\sum_{\mu=1}^{4}\gamma^{\mu}[U_{\mu}(x)\psi_n(x+\hat{\mu}) - U_{-\mu}(x)\psi_n(x-\hat{\mu})] = \lambda_n\psi_n(x)$$

Gauge trans. property:

$$U_{\mu}(x) \to V(x)U_{\mu}(x)V^{+}(x+\hat{\mu})$$

$$\psi_n(x) \to V(x)\psi_n(x)$$

same as quark field apart from an irrelevant phase factor

$$\langle m|n\rangle = \int d^4x\,\psi_m^{+}(x)\psi_n(x) = \delta_{mn}$$:normalization

To keep the gauge symmetry manifestly,
we take the following "operator formalism".

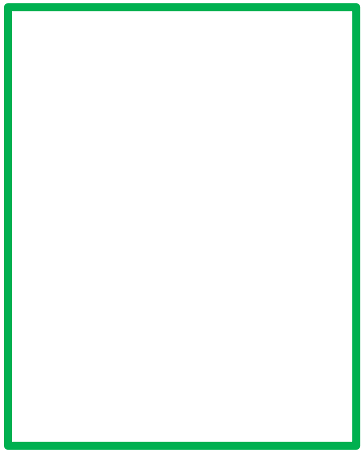
- Link-variable operator $\hat{U}_\mu$ is defined by the matrix element of

$$\langle x | \hat{U}_\mu | y \rangle = U_\mu(x)\delta_{x+\hat{\mu},y}$$

- Wilson Loop operator $\hat{W}$ is defined as the product of $\hat{U}_\mu$ along a rectangular loop:

$$\hat{W} \equiv \prod_{k=1}^{L} \hat{U}_{\mu_k} = \hat{U}_{\mu_1}\hat{U}_{\mu_2}\cdots\hat{U}_{\mu_L}$$

For loops $\sum_{k=1}^{L} \mu_k = 0$



rectangular loop

Functional Trace of Wilson Loop operator is proportional to ordinary vacuum expectation value of the Wilson loop

- Wilson Loop operator: $\hat{W} \equiv \prod_{k=1}^{L} \hat{U}_{\mu_k} = \hat{U}_{\mu_1} \hat{U}_{\mu_2} \cdots \hat{U}_{\mu_L}$

- Functional Trace of Wilson Loop operator:

$$
\begin{aligned}
\mathrm{Tr}\hat{W} &= \mathrm{tr}\sum_x \langle x|\hat{W}|x\rangle = \mathrm{tr}\sum_x \langle x|\hat{U}_{\mu_1}\hat{U}_{\mu_2}\cdots\hat{U}_{\mu_L}|x\rangle \\
&= \mathrm{tr}\sum_{x_1,x_2,\ldots,x_L} \langle x_1|\hat{U}_{\mu_1}|x_2\rangle\langle x_2|\hat{U}_{\mu_2}|x_3\rangle\langle x_3|\hat{U}_{\mu_3}|x_4\rangle\cdots\langle x_L|\hat{U}_{\mu_L}|x_1\rangle \\
&= \mathrm{tr}\sum_x \langle x|\hat{U}_{\mu_1}|x+\mu_1\rangle\langle x+\mu_1|\hat{U}_{\mu_2}|x+\sum_{k=1}^{2}\mu_k\rangle\cdots\langle x+\sum_{k=1}^{L-1}\mu_k|\hat{U}_{\mu_L}|x\rangle \\
&= \mathrm{tr}\sum_x U_{\mu_1}(x)U_{\mu_2}(x+\mu_1)U_{\mu_3}(x+\sum_{k=1}^{2}\mu_k)\cdots U_{\mu_L}(x+\sum_{k=1}^{L-1}\mu_k) \\
&= \langle W\rangle\cdot\mathrm{Tr}1
\end{aligned}
$$

Tr : functional trace $\quad$ tr :trace over SU(3) color index

Dirac-mode matrix elements of Link-variable operator:

$$\langle m|\hat{U}_\mu|n\rangle = \sum_x \langle m|x\rangle\langle x|\hat{U}_\mu|x+\hat{\mu}\rangle\langle x+\hat{\mu}|n\rangle = \sum_x \psi_m^+(x) U_\mu(x)\psi_n(x+\hat{\mu})$$

Huge matrix elements :calculable & ***Gauge Invariant***

Gauge transformation:

$$\begin{cases} U_\mu(x) \to V(x)U_\mu(x)V^+(x+\hat{\mu}) \\ \psi_n(x) \to V(x)\psi_n(x) \end{cases}$$

(same as quark field) $\because \sum_y \not{D}_{xy}^{\mathrm{lat}}[U]\psi_n(y) = \lambda_n \psi_n(x)$

Gauge invariance of the Dirac-mode matrix element $\langle m|\hat{U}_\mu|n\rangle$

$$\langle m|\hat{U}_\mu|n\rangle = \sum_x \psi_m^+(x) U_\mu(x)\psi_n(x+\hat{\mu})$$

$$\to \sum_x \psi_m^+(x)V(x)\cdot V^+(x)U_\mu(x)V^+(x+\hat{\mu})\cdot V^+(x+\hat{\mu})\psi_n(x+\hat{\mu})$$

$$= \sum_x \psi_m^+(x) U_\mu(x)\psi_n(x+\hat{\mu}) = \langle m|\hat{U}_\mu|n\rangle$$

apart from an irrelevant phase factor

Dirac-mode expansion and projection

$$\sum_n |n\rangle\langle n| = 1$$:completeness of the Dirac-mode basis

$$\hat{U}_\mu \equiv \sum_m \sum_n |m\rangle\langle m|\hat{U}_\mu|n\rangle\langle n|$$ Dirac-mode expansion

We define Projection operator which restricts the Dirac-mode space.

Projection operator $$\hat{P} \equiv \sum_{n\in A} |n\rangle\langle n|$$ $\hat{P}^2 = \hat{P}$ $\hat{P}^+ = \hat{P}$

In this projection, the Dirac-mode sum is done within a subset *A*.

e.g. IR-cut $$\sum_{n\in A} = \sum_{|n|>N_{\mathrm{IR}}}$$

⇒ Projected Link-variable operator

$$\hat{U}_\mu^P \equiv \hat{P}\hat{U}_\mu\hat{P} = \sum_{m\in A}\sum_{n\in A} |m\rangle\langle m|\hat{U}_\mu|n\rangle\langle n|$$

- Wilson Loop operator: $\hat{W} \equiv \prod_{k=1}^{L} \hat{U}_{\mu_k} = \hat{U}_{\mu_1} \hat{U}_{\mu_2} \cdots \hat{U}_{\mu_L}$

- Dirac-mode projected Wilson Loop operator:

$$\hat{W}^P \equiv \prod_{k=1}^{L} \hat{U}^P_{\mu_k} = \hat{U}^P_{\mu_1} \hat{U}^P_{\mu_2} \cdots \hat{U}^P_{\mu_L} = \hat{P}\hat{U}_{\mu_1}\hat{P}\hat{U}_{\mu_2}\hat{P}\cdots\hat{P}\hat{U}_{\mu_L}\hat{P}$$

$$= \sum_{n_1, n_2, \ldots, n_{L+1} \in A} |n_1\rangle\langle n_1|\hat{U}_{\mu_1}|n_2\rangle\langle n_2|\hat{U}_{\mu_2}|n_3\rangle\cdots\langle n_L|\hat{U}_{\mu_L}|n_{L+1}\rangle\langle n_{L+1}|$$

- Dirac-mode projected Wilson Loop:

$$\mathrm{Tr}\hat{W}^P \equiv \mathrm{Tr}\prod_{k=1}^{L} \hat{U}^P_{\mu_k} = \mathrm{Tr}\hat{U}^P_{\mu_1} \hat{U}^P_{\mu_2} \cdots \hat{U}^P_{\mu_L} = \mathrm{Tr}\,\hat{P}\hat{U}_{\mu_1}\hat{P}\hat{U}_{\mu_2}\hat{P}\cdots\hat{P}\hat{U}_{\mu_L}\hat{P}$$

$$= \sum_{n_1, n_2, \ldots, n_L \in A} \mathrm{tr}\,\langle n_1|\hat{U}_{\mu_1}|n_2\rangle\langle n_2|\hat{U}_{\mu_2}|n_3\rangle\cdots\langle n_L|\hat{U}_{\mu_L}|n_1\rangle$$

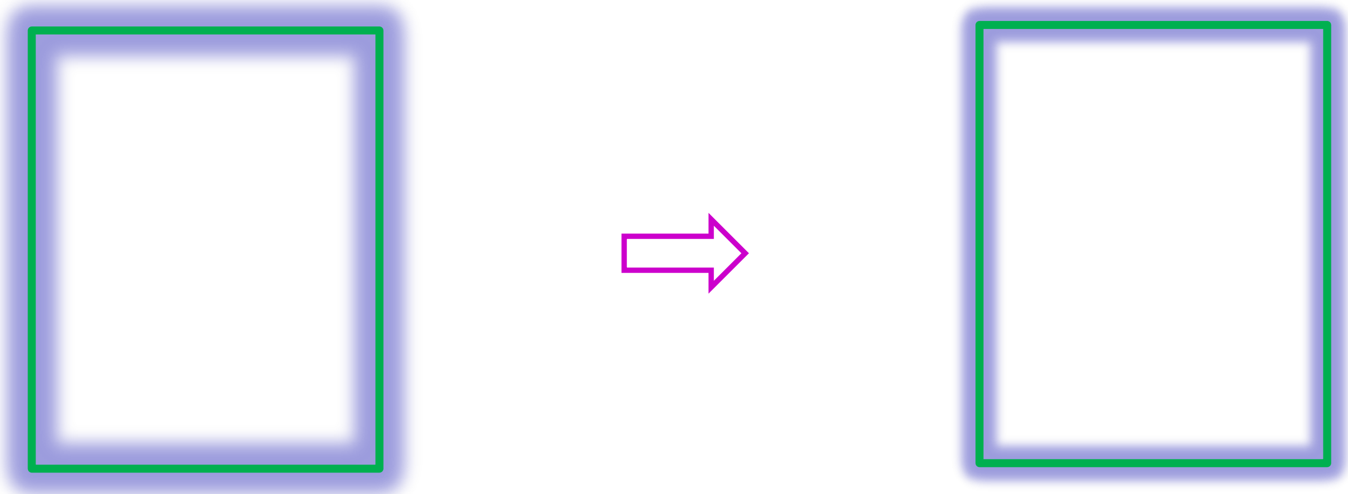
Gauge Invariant !

tr: color index

Its Gauge Invariance is also checked in lattice QCD calculation.

Dirac-mode projected Wilson Loop

$$\mathrm{Tr}\hat{W}^P \equiv \mathrm{Tr}\prod_{k=1}^{L}\hat{U}^P_{\mu_k} = \sum_{n_1,n_2,\ldots,n_L\in A}\mathrm{tr}\,\langle n_1|\hat{U}_{\mu_1}|n_2\rangle\langle n_2|\hat{U}_{\mu_2}|n_3\rangle\cdots\langle n_L|\hat{U}_{\mu_L}|n_1\rangle$$



The original Wilson loop couples to all the Dirac modes and obeys the area law..

The projected Wilson loop couples to restricted Dirac modes.

Based on this relation, we investigate the role of specific Dirac modes to the area law of the Wilson loop. In fact, if some Dirac modes are essential to reproduce the area law or the confinement property, the removal of the coupling to these modes leads to a significant change on the area law.

Dirac-mode projected Inter-Quark Potential

Dirac-mode projected Wilson Loop

$$\mathrm{Tr}\hat{W}^P \equiv \mathrm{Tr}\prod_{k=1}^{L}\hat{U}^P_{\mu_k} = \sum_{n_1,n_2,\ldots,n_L\in A}\mathrm{tr}\,\langle n_1|\hat{U}_{\mu_1}|n_2\rangle\langle n_2|\hat{U}_{\mu_2}|n_3\rangle\cdots\langle n_L|\hat{U}_{\mu_L}|n_1\rangle$$

⇒ corresponding Potential

$$V^P(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\left\{\mathrm{Tr}\hat{W}^P(R,T)\right\}$$

As a caution, some non-locality appears.

Unprojected case: ordinary inter-quark potential is obtained

cf Trace of Wilson Loop operator is proportional to ordinary vacuum expectation value of the Wilson loop

$$\mathrm{Tr}\hat{W} = \langle W\rangle\cdot\mathrm{Tr}1$$

$$V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\left\{\mathrm{Tr}\hat{W}(R,T)\right\} = -\lim_{T\to\infty}\frac{1}{T}\ln\langle W(R,T)\rangle + \text{irrelevant const.}$$

As a technical demerit of this formalism, we have to deal with Huge dimensional matrix and their products.

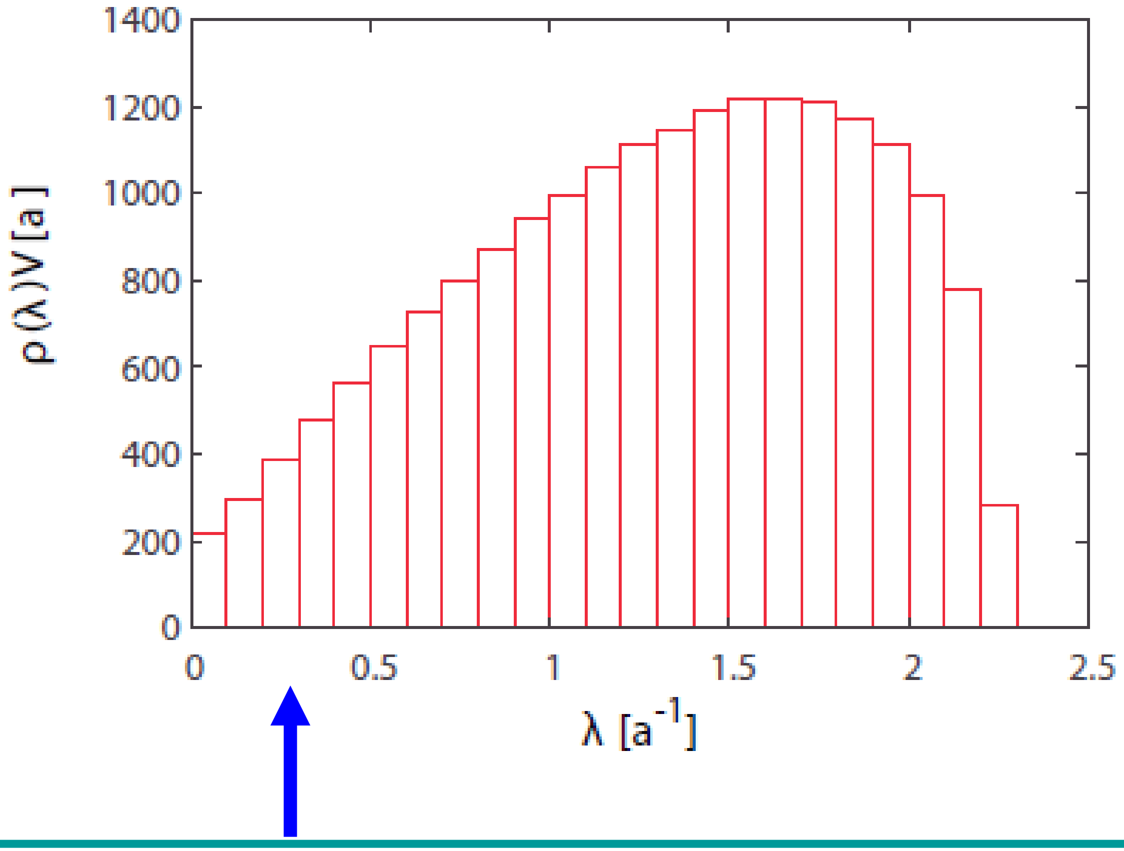
Actually, for the matrix $\langle m|\hat{U}_{\mu_1}|n\rangle$,
the total matrix dimension is (Dirac-mode number)^2.
Here, the Dirac-mode number is (lattice-volume) x N_c x 4.
This number can be reduced to be (lattice-volume) x N_c,
using the Kogut-Susskind technique.

At present, we use a small-size lattice in this calculation.

Lattice Calculation Condition:
SU(3) plaquette action on quenched periodic lattice
β=5.6 (i.e., a=0.25fm) , 6^4

Eigen-value distribution of QCD Dirac operator

β=5.6 (a=0.25fm for lattice spacing), 6^4 lattice

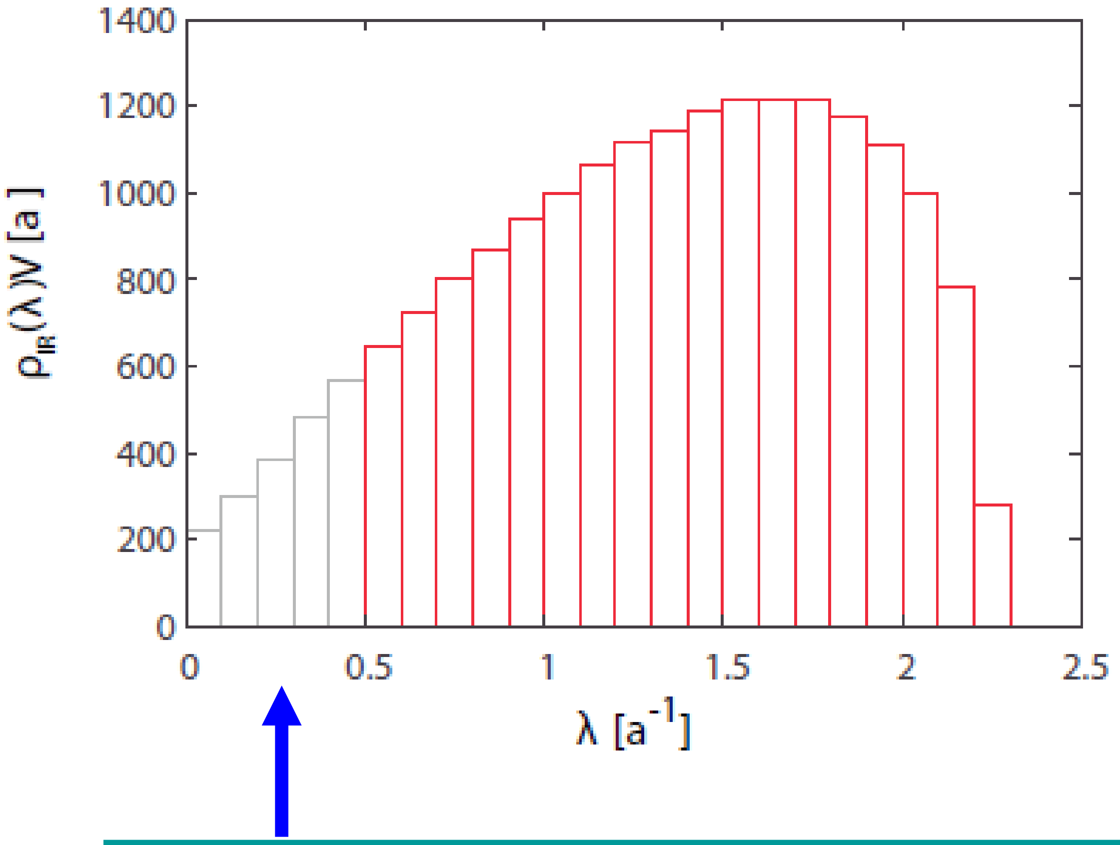


Low-lying Dirac modes are responsible to Chiral Symmetry Breaking

(cf. Banks-Casher relation)

Eigen-value distribution of QCD Dirac operator

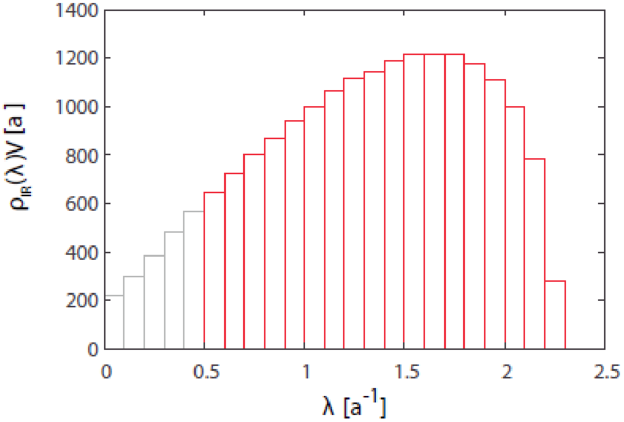
β=5.6 (a=0.25fm for lattice spacing), 6^4 lattice



By Removing the Low-lying Dirac modes,
Chiral Condensate is Largely Reduced.

(cf. Banks-Casher relation)

Chiral Condensate after removing low-lying Dirac modes



$$\langle \bar{q}q \rangle_{IR} \propto \sum_{\lambda_n \geq \Lambda_{IR}} \frac{2m}{\lambda_n^2 + m^2}$$

$$\frac{\langle \bar{q}q \rangle_{IR}}{\langle \bar{q}q \rangle} \approx 0.02$$

for m_q~5 MeV

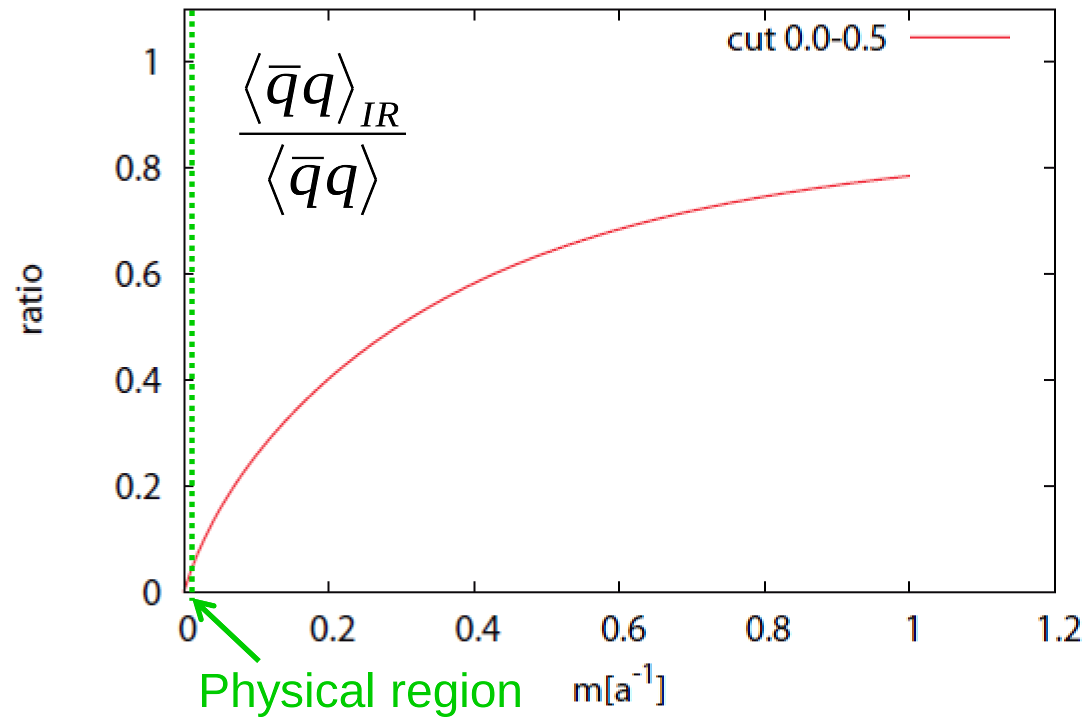
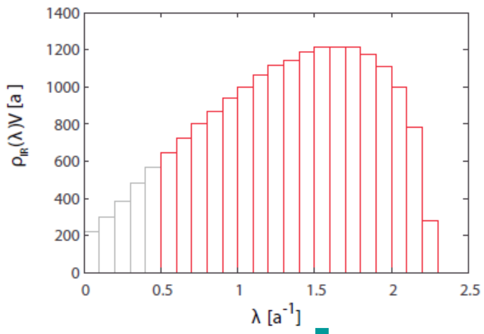


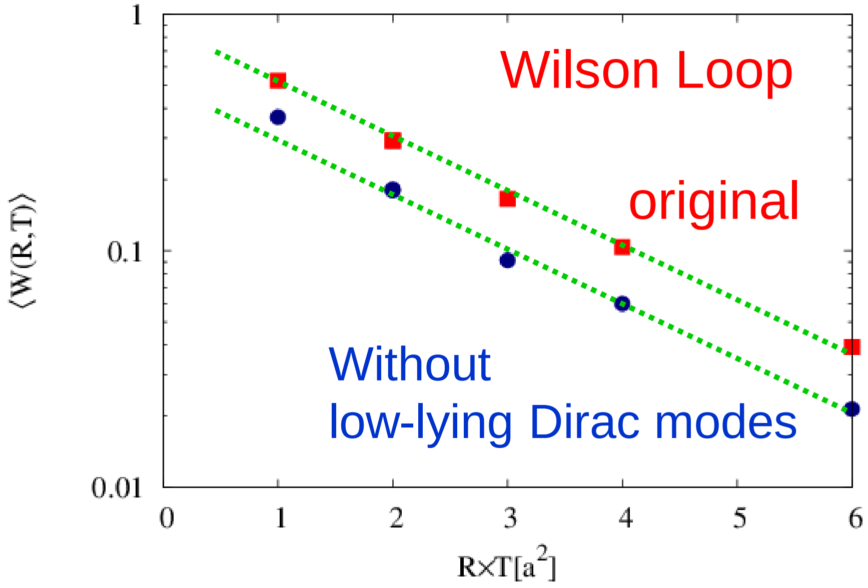
FIG. 2: The lattice QCD result of the quark condensate $\langle \bar{q}q \rangle_{\Lambda_{IR}}$ as the function of the current quark mass m in the presence of IR cut $\Lambda_{IR} = 0.5, 1.0, 1.5[a^{-1}]$. The vertical axis is normalized by the original value of $\langle \bar{q}q \rangle$ without cut. A large reduction is found as $\langle \bar{q}q \rangle_{\Lambda_{IR}} / \langle \bar{q}q \rangle \simeq 0.02$ for $\Lambda_{IR} = 0.5a^{-1} \simeq 0.4$GeV around the physical region of $m \simeq 0.006a^{-1} \simeq 5$MeV.

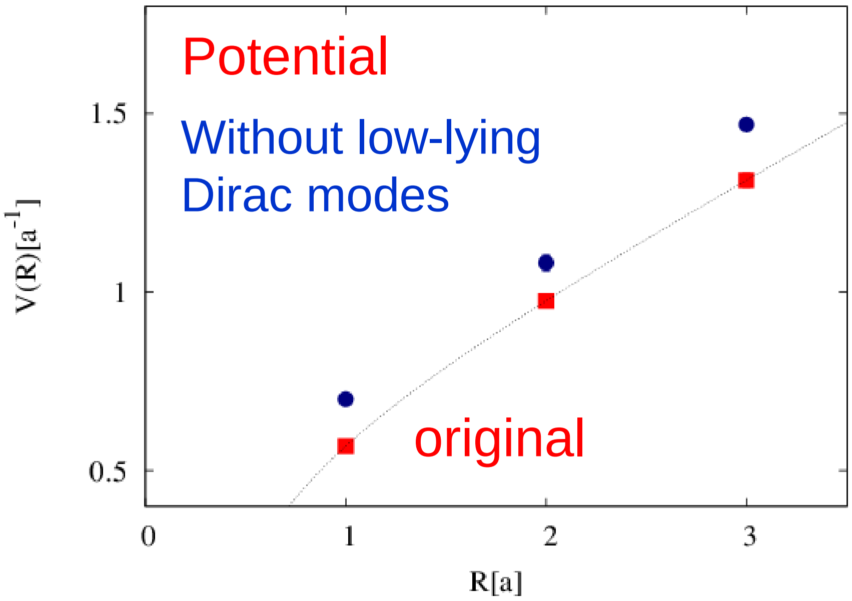
Chiral Condensate is largely reduced (only 2%!)
after removing the low-lying Dirac modes.

Wilson Loop after removing low-lying Dirac modes



Lattice QCD result of
Wilson Loop and Inter-Quark Potential
after removing low-lying Dirac modes





Wilson Loop obeys the Area law with the same slope even after removing the low-lying Dirac modes, which are responsible to chiral symmetry breaking.

Dirac-mode projected Polyakov Loop and Z_3 Center Symmetry

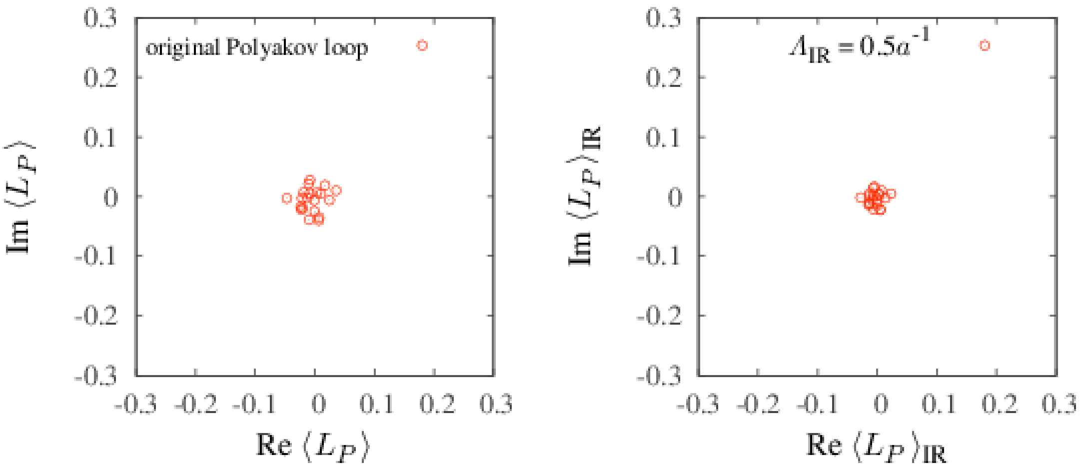
Dirac-mode projected Polyakov Loop

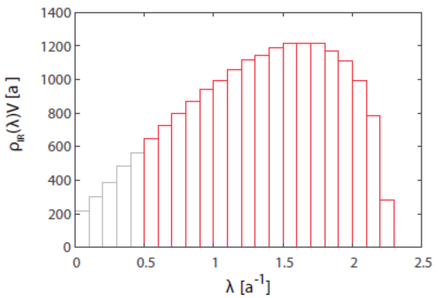
$$\mathrm{Tr}\hat{P}^P \equiv \mathrm{Tr}(\hat{U}_4^P)^T = \sum_{n_1,n_2,\ldots,n_T \in A} \mathrm{tr}\,\langle n_1|\hat{U}_4|n_2\rangle\langle n_2|\hat{U}_4|n_3\rangle\cdots\langle n_T|\hat{U}_4|n_1\rangle$$

Polyakov Loop

Without IR-Dirac modes

on periodic lattice





Dirac spectral density

FIG. 6: The scatter plot of the Polyakov loop. The left figure shows the original Polyakov loop $\langle L_P \rangle$. The right figure shows the Polyakov loop $\langle L_P \rangle_{\mathrm{IR}}$ after cutting off the low-lying Dirac modes below the IR-cutoff $\Lambda_{\mathrm{IR}} = 0.5a^{-1}$.

Even after removing the low-lying Dirac modes, Polyakov loop remains to be zero, which means confinement phase and unbroken Z_3-center symmetry.

UV-cut case of Dirac modes

Dirac spectral density Wilson Loop Potential

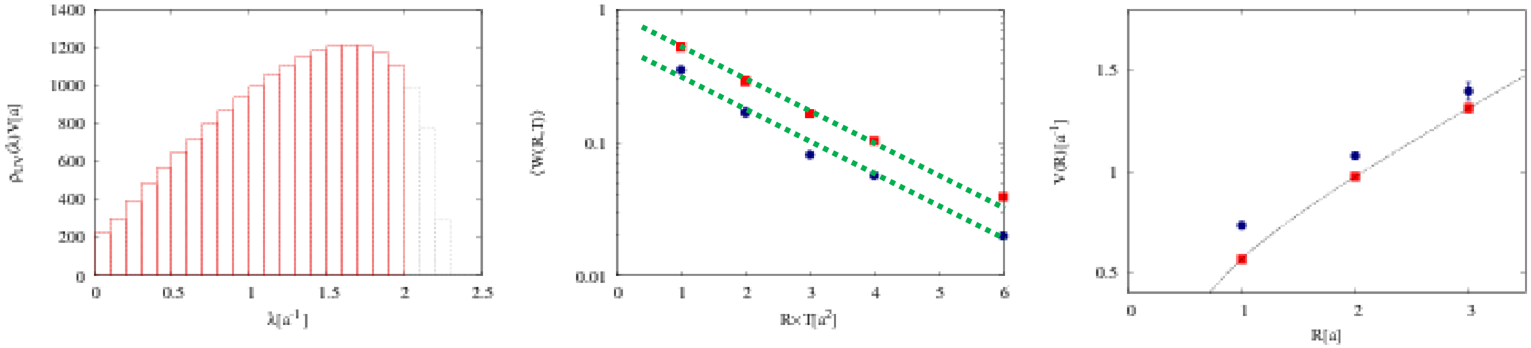


FIG. 7: (a) The UV-cut Dirac spectral density $\rho_{\mathrm{UV}}(\lambda) \equiv \rho(\lambda)\theta(\Lambda_{\mathrm{UV}} - |\lambda|)$ with the UV-cutoff $\Lambda_{\mathrm{UV}} = 2a^{-1} \simeq 1.6\mathrm{GeV}$. (b) The UV-cut Wilson loop $\mathrm{Tr}W^P(R,T)$ (circle) after removing the UV Dirac modes, plotted against $R \times T$. The slope parameter σ^P is almost the same as that of the original Wilson loop (square). (c) The corresponding UV-cut inter-quark potential (circle), which is almost unchanged from the original one (square), apart from an irrelevant constant.

Wilson Loop obeys the Area law with the same slope after removing the UV Dirac modes.

Intermediate-cut cases of Dirac modes

Dirac spectral density | Wilson Loop | Potential

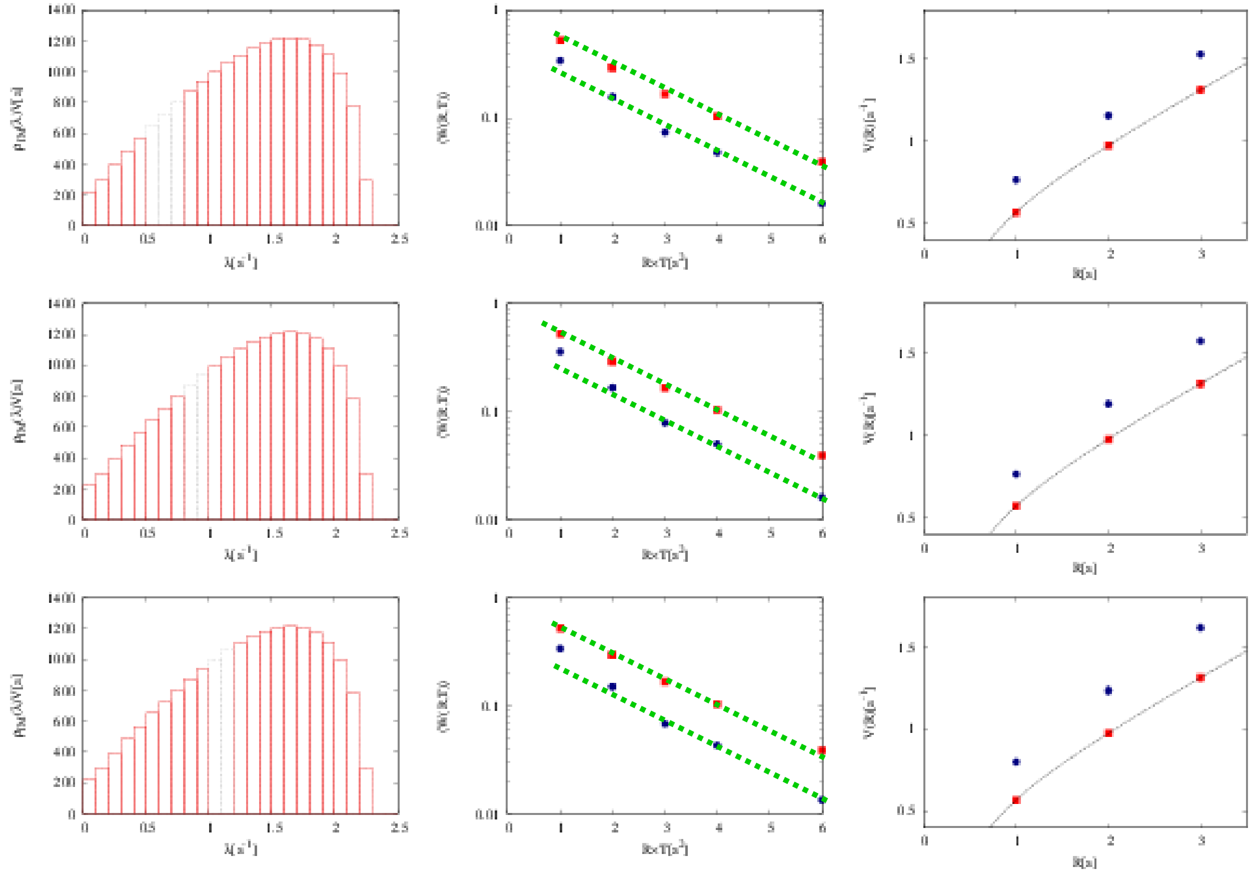


FIG. 8: The left figures show the intermediate(IM)-cut Dirac spectral density $\rho_{\rm IM}(\lambda)$: the IM Dirac modes of $0.5-0.8[a^{-1}]$ (top), $0.8-1.0[a^{-1}]$ (middle), and $1.0-1.2[a^{-1}]$ (bottom) are cut. The central figures show the IM-cut Wilson loop $\mathrm{Tr}W^{P}(R,T)$ (circle) after removing the IM Dirac modes, plotted against $R\times T$. For each case, the slope parameter σ^{P} is almost the same as that of the original Wilson loop (square). The right figure shows the corresponding IM-cut inter-quark potential (circle), which is almost unchanged from the original one (square), apart from an irrelevant constant.

Wilson Loop obeys the Area law with the same slope after removing various Dirac modes.

Related Lattice Studies

F.Synatschke, A.Wipf, and K.Langfeld, Phys. Rev. D77, 114018 (2008).

They found that confining force are reproduced with low-lying Dirac modes.

Our comment: Their result seems to be consistent with our result on UV-cut case of Dirac modes.

C.B.Lang and M.Schrock,
Phys. Rev. D84, 087704 (2011); PoS (LAT2011), 111 (2011);
L.Ya Glozman, C.B.Lang and M.Schrock, arXiv:1205.4887 (2012).

They studied Hadron Spectra after cutting off the low-lying Dirac modes.

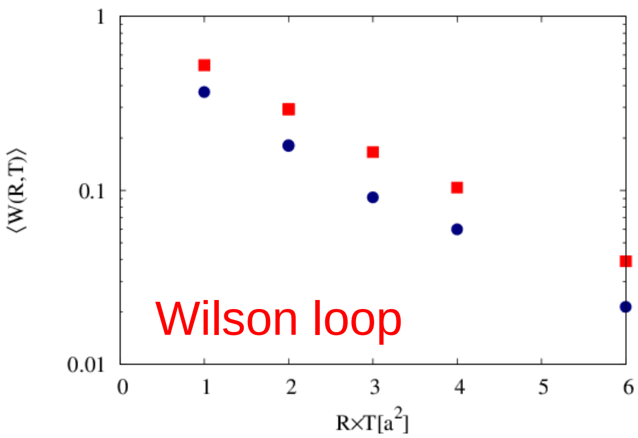
Our comment: The hadron formation seems to indicate the existence of Confinement Force.

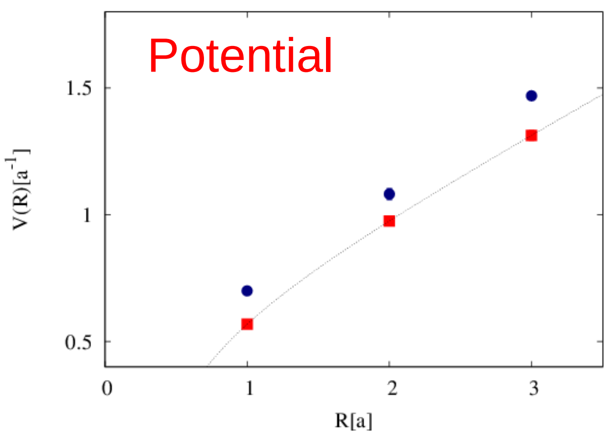
Summary and Concluding Remarks

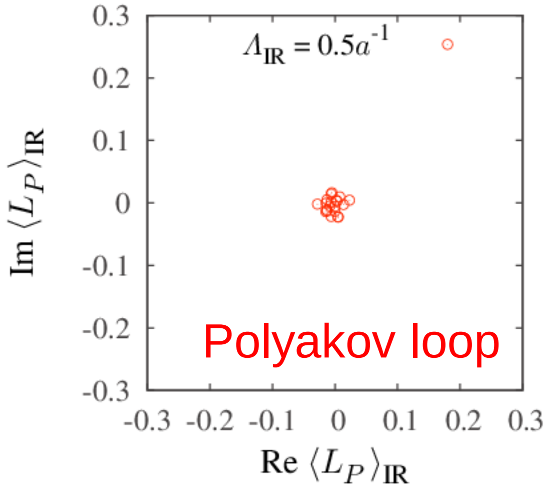
With the Dirac-mode expansion, we have analyzed relation between confinement and CSB in SU(3) lattice QCD.

Even after removing the low-lying Dirac modes,
which are responsible to chiral symmetry breaking,
Wilson loop obeys the Area law with the same slope parameter,
and Polyakov loop remains to be zero, which means
the confinement phase and unbroken Z_3-center symmetry.

These indicate that one-to-one correspondence does not hold for between confinement and chiral symmetry breaking in QCD.







Instanton and Dirac zero-mode

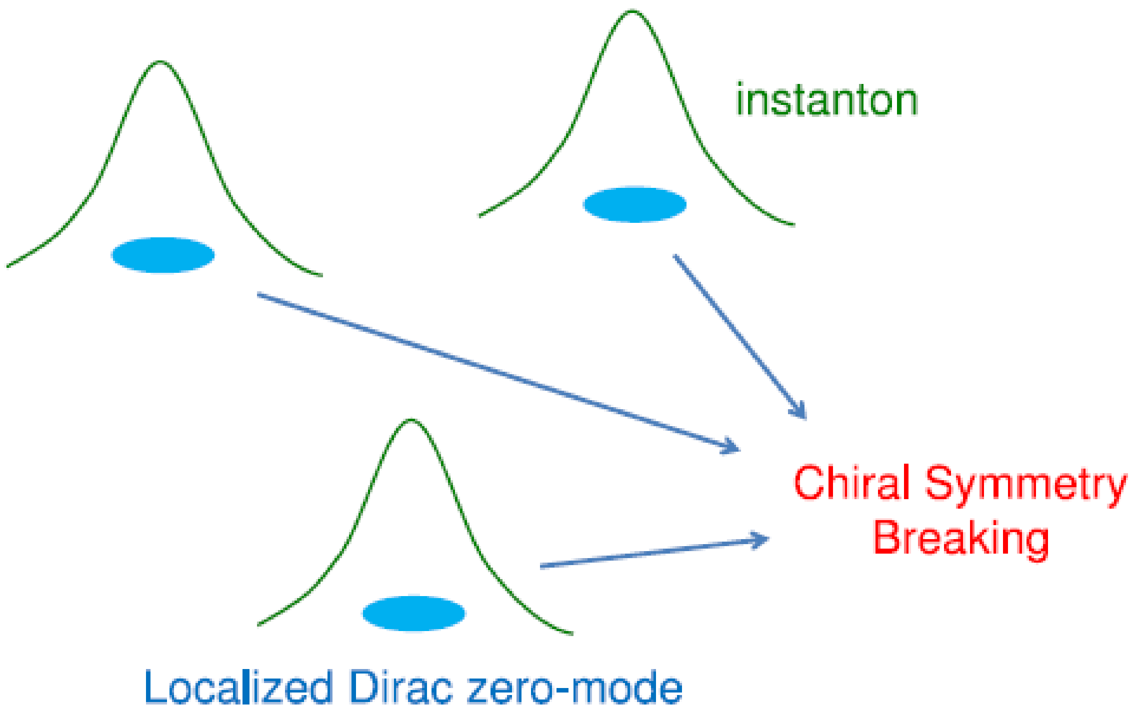


Figure 8: Around each instanton, the Dirac zero-mode is localized, and such low-lying Dirac modes contribute to chiral symmetry breaking. However, the localized objects are hard to contribute to confinement.

Recall that instantons contribute to chiral symmetry breaking, but do not directly lead to confinement [8]. Then, as a thought experiment, if only instantons can be carefully removed from the QCD vacuum, confinement properties would be almost unchanged, but the chiral condensate is largely reduced, and accordingly some low-lying Dirac modes disappear. Thus, in this case, confinement is almost unchanged, in spite of the large reduction of low-lying Dirac modes.