The effects of flavour symmetry breaking on hadron matrix elements I

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Project

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Bietenholz et al; Phys. Rev. D 84, 054509 (2011);
arXiv:1102.5300 [hep-lat]
arXiv:1012.4371 [hep-lat] (Lat10)
Phys.Lett.B690:436-441,2010; arXiv:1003.1114 [hep-lat]

Abstract

The QCD interaction is flavour-blind. Neglecting electromagnetic and weak interactions, the only difference between flavours comes from the mass matrix. We investigate how flavour-blindness constrains hadron masses and matrix elements after flavour SU(3) is broken by the mass difference between the strange and light quarks, to help us extrapolate 2+1 flavour lattice data to the physical point.

We have our best theoretical understanding when all 3 quark flavours have the same masses (because we can use the full power of flavour $SU(3)$); nature presents us with just one instance of the theory, with $m_s/m_l \approx 25.$ We are interested in interpolating between these two cases.

Lattice

- On the lattice we can choose our quark masses, so we can investigate fictional universes where $m_s/m_l \neq 25$, and so gain a clearer understanding of symmetry breaking
- Computational expense gets too high if the u and d quarks are too light. We still need to extrapolate to get from the lattice results to the physical point.

Introduction

Split QCD action into a flavour-symmetric part and a small perturbation.

$$\begin{aligned}\text{Large Piece} &= \text{Kinetic Terms}\\ &\quad + \text{Gluon-Gluon Vertices}\\ &\quad + \text{Quark-Gluon Vertices}\\ &\quad + \text{Singlet Quark Mass Term}\end{aligned}$$

$$\text{Small Piece} = \text{Non-Singlet Quark Mass Terms}$$

Long history: M. Gell Man, Phys Rev 125 (1962) 1067.
S. Okubo, Prog Theor Phys 27 (1962) 949.

Quark Masses

Notation

$$
\begin{aligned}
\text{Fix} \qquad \overline{m} &\equiv \frac{1}{3}(m_u + m_d + m_s) \\
\delta m_u &\equiv m_u - \overline{m} \\
\delta m_d &\equiv m_d - \overline{m} \\
\delta m_s &\equiv m_s - \overline{m} \\
\\
\delta m_u + \delta m_d + \delta m_s &= 0 \\
\\
m_l &\equiv \frac{1}{2}(m_u + m_d) \\
\delta m_l &\equiv m_l - \overline{m}
\end{aligned}
$$

Quark Masses

The quark mass matrix is

$$\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} = \overline{m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}(\delta m_u - \delta m_d) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\delta m_s \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\mathcal{M}$ has a flavour singlet part (proportional to I) and a flavour octet part, proportional to λ_3, λ_8.

$SU(3)$ **breaking**

How severely does the strange quark break $SU(3)$ symmetry?
In this approach it is not the strange-light ratio, $m_s/m_l \sim 25$, that matters.
A more natural way to judge the severity of symmetry breaking is to compare $m_s - m_l$ with a typical hadronic mass,

$$m_s - m_l \ll M_H$$

Because $m_s - m_l$ is small in this sense, can hope for an expansion with good convergence.

Strategy

Start from a point with all 3 sea quark masses equal,

$$m_u = m_d = m_s \equiv m_0$$

and extrapolate towards the physical point, keeping the average sea quark mass

$$\overline{m} \equiv \frac{1}{3}(m_u + m_d + m_s)$$

constant.
Starting point has

$$m_0 \approx \frac{1}{3} m_s^{phys}$$

As we approach the physical point, the u and d become lighter, but the s becomes heavier. Pions are decreasing in mass, but K and η increase in mass as we approach the physical point.

Flavour Hierarchy

$$\begin{aligned}\text{Large Piece} \quad &= \quad \text{Kinetic Terms} \\ &\quad + \text{Gluon-Gluon Vertices} \\ &\quad + \text{Quark-Gluon Vertices} \\ &\quad + \text{Singlet Quark Mass Term}\end{aligned}$$

$$\text{Small Piece} \quad = \quad \text{Non-Singlet Quark Mass Terms}$$

All terms in Large Piece are flavour singlets, leave $SU(3)$ unbroken.
Small Piece is pure flavour octet.
Higher $SU(3)$ representations completely absent from QCD action.

Flavour Hierarchy

Higher representations of $SU(3)$ are absent from the QCD action, but they appear at higher orders in the perturbation. Square an octet — generates 27-plet.

δm_q^0	1					
δm_q^1		8				
δm_q^2	1	8			27	
δm_q^3	1	8	10	$\overline{10}$	27	64

SU(3) classification

- Classify physical quantities by $SU(3)$ and sub-group $SU(2)$.
- Classify quark mass polynomials in same way.
- Quantity of Known Symmetry $=$ Polynomials of Matching Symmetry
- Taylor expansion about (m_0, m_0, m_0) strongly constrained by symmetry.

Matrix Elements

In this talk we are investigating non-singlet matrix elements acting between octet baryons, (e.g. V and A currents for weak decays).

$\langle H^i | O^j | H^k \rangle$

$i, j, k \in \{1, \cdots, 8\}$

Amplitudes form an $8 \times 8 \times 8$ tensor.

Need the $SU(3)$ classification of $8 \times 8 \times 8$ tensors, in just the same way as we needed to give the classification of 8×8 and 10×10 matrices for baryon masses.

Matrix Elements

Index	Baryon	Meson	Operator
1	n	K^0	$\bar{d}\Gamma s$
2	p	K^+	$\bar{u}\Gamma s$
3	Σ^-	π^-	$\bar{d}\Gamma u$
4	Σ^0	π^0	$\frac{1}{\sqrt{2}}\left(\bar{u}\Gamma u - \bar{d}\Gamma d\right)$
5	Λ^0	η	$\frac{1}{\sqrt{6}}\left(\bar{u}\Gamma u + \bar{d}\Gamma d - 2\bar{s}\Gamma s\right)$
6	Σ^+	π^+	$\bar{u}\Gamma d$
7	Ξ^-	K^-	$\bar{s}\Gamma u$
8	Ξ^0	$\bar{K}^0$	$\bar{s}\Gamma d$

Matrix Elements

Use the corresponding meson name to label the octet operators,

$$\langle p|\pi^0|p\rangle \equiv \langle p|\frac{1}{\sqrt{2}}(\bar{u}\Gamma u - \bar{d}\Gamma d)|p\rangle \equiv \langle H^2|O^4|H^2\rangle$$

Matrix Elements

To find the allowed mass-dependence of octet matrix elements of octet hadrons we need the $SU(3)$ decomposition of $8 \otimes 8 \otimes 8$. Using the intermediate result

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$$

we find

$$\begin{aligned} 8 \otimes 8 \otimes 8 \;=\; & 1 \oplus 1 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \\ & \oplus 27 \oplus 27 \oplus 27 \oplus 27 \oplus 27 \oplus 27 \oplus 64 \\ & \oplus 10 \oplus 10 \oplus 10 \oplus 10 \oplus \overline{10} \oplus \overline{10} \oplus \overline{10} \oplus \overline{10} \\ & \oplus 35 \oplus 35 \oplus \overline{35} \oplus \overline{35} \end{aligned}$$

Matrix Elements

The allowed quark mass Taylor expansion for a hadronic matrix element must follow the schematic pattern

$$
\begin{aligned}
\langle H^i | O^j | H^k \rangle \;&=\; \sum (\text{singlet mass polynomial}) \times (\text{singlet tensor})^{ijk} \\
&+ \sum (\text{octet polynomial}) \times (\text{octet tensor})^{ijk} \\
&+ \sum (\text{27-plet polynomial}) \times (\text{27-plet tensor})^{ijk} \\
&+ \sum (\text{64-plet polynomial}) \times (\text{64-plet tensor})^{ijk} \\
&+ \cdots
\end{aligned}
$$

The tensors in this equation are three-dimensional arrays of integers and square-roots of integers, objects somewhat analogous to three-dimensional Gell-Mann matrices.

Matrix Elements

$$
\begin{aligned}
\langle H^i | O^j | H^k \rangle \quad = \quad & \sum (\text{singlet mass polynomial}) \times (\text{singlet tensor})^{ijk} \\
& + \sum (\text{octet polynomial}) \times (\text{octet tensor})^{ijk} \\
& + \sum (\text{27-plet polynomial}) \times (\text{27-plet tensor})^{ijk} \\
& + \sum (\text{64-plet polynomial}) \times (\text{64-plet tensor})^{ijk} \\
& + \cdots
\end{aligned}
$$

Mass polynomials with the symmetry $10, \overline{10}, 35, \overline{35}$ all have factors of $(m_u - m_d)$. So they only appear if we consider the $1+1+1$ case of symmetry breaking. At present we are only considering the $2+1$ case, $m_u = m_d \neq m_s$ so we can forget about the $10, \overline{10}, 35, \overline{35}$ representations.

Matrix Elements

We found just two singlet tensors in the expansion of $8 \otimes 8 \otimes 8$, so at the symmetric point there are only two independent coefficients (usually called F, D or f, d) needed to completely specify all the matrix elements between the members of the octet.

These give the classic $SU(3)$ inter-relations between octet amplitudes. These are generally found to work rather well. We should however be able to do better by also including higher terms in the mass expansion.

There are 8 octets in the expansion of $8 \otimes 8 \otimes 8$, so if we work to first order δm_q, the $SU(3)$ violation, we have 8 new coefficients.

There are still many fewer coefficients than there are amplitudes, so there are numerous constraints and cross-relations between amplitudes.

Matrix Elements

		1		8							
I	$A_{\bar{B}MB}$	f	d	g_1	g_2	g_3	g_4	g_5	h_1	h_2	h_3
0	$\overline{N}\eta N$	$\sqrt{3}$	-1	1	0	0	0	0	0	-1	0
0	$\overline{\Sigma}\eta\Sigma$	0	2	1	0	$2\sqrt{3}$	0	0	0	0	0
0	$\overline{\Lambda}\eta\Lambda$	0	-2	1	2	0	0	0	0	0	0
0	$\overline{\Xi}\eta\Xi$	$-\sqrt{3}$	-1	1	0	0	0	0	0	1	0
1	$\overline{N}\pi N$	1	$\sqrt{3}$	0	0	-2	0	0	2	0	0
1	$\overline{\Sigma}\pi\Sigma$	2	0	0	0	0	0	0	-2	$\sqrt{3}$	0
1	$\overline{\Xi}\pi\Xi$	1	$-\sqrt{3}$	0	0	2	0	0	2	0	0
1	$\overline{\Sigma}\pi\Lambda$	0	2	0	1	$-\sqrt{3}$	i	0	0	0	0
$\frac{1}{2}$	$\overline{N}K\Sigma$	$-\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	0	$i\sqrt{2}$	$\sqrt{2}$	0	$i\sqrt{6}$
$\frac{1}{2}$	$\overline{N}K\Lambda$	$-\sqrt{3}$	-1	0	1	0	i	$i\sqrt{3}$	$-\sqrt{3}$	1	$-i$
$\frac{1}{2}$	$\overline{\Lambda}K\Xi$	$\sqrt{3}$	-1	0	1	0	$-i$	$-i\sqrt{3}$	$\sqrt{3}$	-1	$-i$
$\frac{1}{2}$	$\overline{\Sigma}K\Xi$	$\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	0	$-i\sqrt{2}$	$-\sqrt{2}$	0	$i\sqrt{6}$

Matrix Elements

		1		8							
I	$A_{\bar{B}MB}$	f	d	g_1	g_2	g_3	g_4	g_5	h_1	h_2	h_3
0	$\overline{N}\eta N$	$\sqrt{3}$	-1	1	0	0	0	0	0	-1	0
0	$\overline{\Sigma}\eta\Sigma$	0	2	1	0	$2\sqrt{3}$	0	0	0	0	0
0	$\overline{\Lambda}\eta\Lambda$	0	-2	1	2	0	0	0	0	0	0
0	$\overline{\Xi}\eta\Xi$	$-\sqrt{3}$	-1	1	0	0	0	0	0	1	0

$$\vdots$$

Table gives amplitudes for $\{p, \Lambda^0, \Sigma^+, \Xi^0\}$

$$\begin{aligned} \langle p|\eta|p\rangle = A_{\overline{N}\eta N} &= \sqrt{3}f - d + (g_1 - h_2)\delta m_l \\ \langle \Sigma^+|\eta|\Sigma^+\rangle = A_{\overline{\Sigma}\eta\Sigma} &= 2d + (g_1 + 2\sqrt{3}g_3)\delta m_l \end{aligned}$$

$$\vdots$$

Matrix Elements

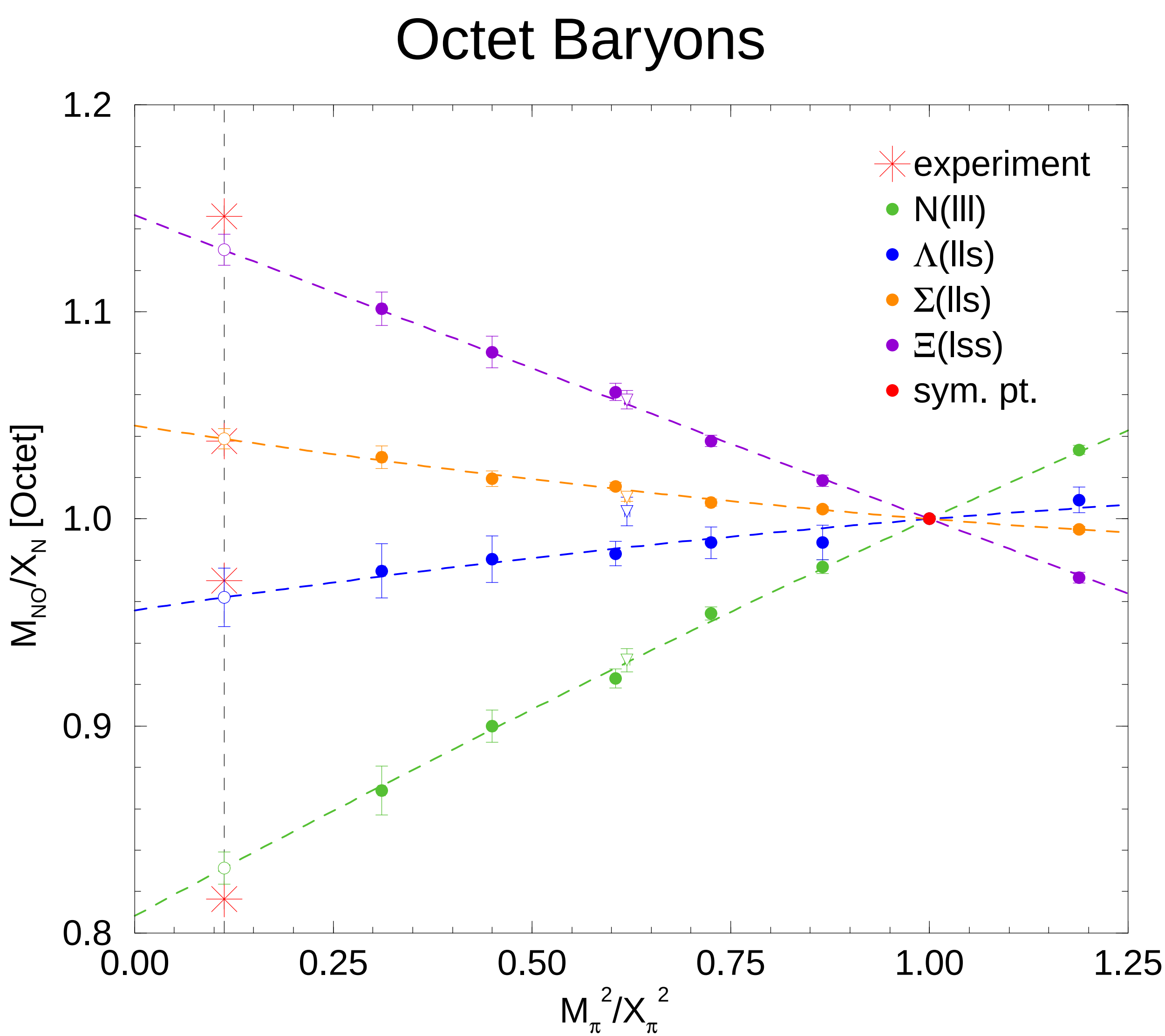
		1		8							
I	$A_{\bar{B}MB}$	f	d	g_1	g_2	g_3	g_4	g_5	h_1	h_2	h_3
0	$\overline{N}\eta N$	$\sqrt{3}$	-1	1	0	0	0	0	0	-1	0
0	$\overline{\Sigma}\eta\Sigma$	0	2	1	0	$2\sqrt{3}$	0	0	0	0	0
0	$\overline{\Lambda}\eta\Lambda$	0	-2	1	2	0	0	0	0	0	0
0	$\overline{\Xi}\eta\Xi$	$-\sqrt{3}$	-1	1	0	0	0	0	0	1	0

$$\vdots$$

Table gives amplitudes for $\{p, \Lambda^0, \Sigma^+, \Xi^0\}$

All other amplitudes follow from these by (unbroken) isospin symmetry.

Fan Plot



Octet Baryons
1.2
1.1
1.0
0.9
0.8
0.00
0.25
0.50
0.75
1.00
1.25
M_{NO}/X_N [Octet]
M_π^2/X_π^2
experiment
N(lll)
Λ(lls)
Σ(lls)
Ξ(lss)
sym. pt.

The d-fan

Using the Table we can construct seven quantities D_i, which all have the same value ($2d$) at the symmetric point, but which can differ once $SU(3)$ is broken.

$$
\begin{aligned}
D_1 \equiv -(A_{\bar{N}\eta N} + A_{\bar{\Xi}\eta\Xi}) &= 2d - 2g_1\delta m_l \\
D_2 \equiv A_{\bar{\Sigma}\eta\Sigma} &= 2d + (g_1 + 2\sqrt{3}g_3)\delta m_l \\
D_3 \equiv \; - A_{\bar{\Lambda}\eta\Lambda} &= 2d - (g_1 + 2g_2)\delta m_l \\
D_4 \equiv \frac{1}{\sqrt{3}}(A_{\bar{N}\pi N} - A_{\bar{\Xi}\pi\Xi}) &= 2d - \frac{4}{\sqrt{3}}g_3\delta m_l \\
D_5 \equiv \Re A_{\bar{\Sigma}\pi\Lambda} &= 2d + (g_2 - \sqrt{3}g_3)\delta m_l \\
D_6 \equiv \frac{1}{\sqrt{6}}\Re(A_{\bar{N}K\Sigma} + A_{\bar{\Sigma}K\Xi}) &= 2d + \frac{2}{\sqrt{3}}g_3\delta m_l \\
D_7 \equiv -\Re(A_{\bar{N}K\Lambda} + A_{\bar{\Lambda}K\Xi}) &= 2d - 2g_2\delta m_l
\end{aligned}
$$

Plotting these quantities gives a fan plot with 7 lines, but only 3 slope parameters (g_1, g_2 and g_3), so the splittings between these observables are highly constrained.

The f-fan

Similarly, we can construct five quantities F_i, which all have the same value ($2f$) at the symmetric point, but which can differ once $SU(3)$ is broken.

$$
\begin{aligned}
F_1 \equiv \frac{1}{\sqrt{3}}(A_{\bar{N}\eta N} - A_{\bar{\Xi}\eta\Xi}) &= 2f - \frac{2}{\sqrt{3}} h_2 \delta m_l \\
F_2 \equiv (A_{\bar{N}\pi N} + A_{\bar{\Xi}\pi\Xi}) &= 2f + 4h_1 \delta m_l \\
F_3 \equiv A_{\bar{\Sigma}\pi\Sigma} &= 2f + (-2h_1 + \sqrt{3}h_2)\delta m_l \\
F_4 \equiv \frac{1}{\sqrt{2}}\Re(A_{\bar{\Sigma}K\Xi} - A_{\bar{N}K\Sigma}) &= 2f - 2h_1 \delta m_l \\
F_5 \equiv \frac{1}{\sqrt{3}}\Re(A_{\bar{\Lambda}K\Xi} - A_{\bar{N}K\Lambda}) &= 2f + \frac{2}{\sqrt{3}}(\sqrt{3}h_1 - h_2)\delta m_l
\end{aligned}
$$

Plotting these quantities gives a fan plot with 5 lines, but only 2 slope parameters (h_1 and h_2), so the splittings between these observables are highly constrained.

Conclusions

- Extrapolating from lattice simulations to the physical quark masses is made much easier by keeping $m_u + m_d + m_s$ constant.
- Flavour SU(3) analysis strongly constrains Taylor expansions in quark masses.