K semileptonic decays form factors with HISQ valence quarks

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1. Introduction

Experimental average, Antonelli et al. (Flavianet), 1005.2323

$$|V_{us}|f_+(0)^{K\to\pi} = 0.2163(\pm 0.23\%)$$

$$f_+(0)^{K\to\pi} : \begin{array}{l} +0.5\% \\ -0.6\% \end{array}$$

RBC/UKQCD, EPJC69(2010)

* Check unitarity in the first row of CKM matrix.

$$\Delta_{CKM} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = -0.0001(6)$$ **M. Antonelli et al**

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Look for new physics effects in the comparison of $|V_{us}|$ from helicity suppressed $K_{\mu 2}$ versus helicity allowed K_{l3}

$$R_{\mu 23} = \left(\frac{f_K / f_\pi}{f_+^{K\pi}(0)} \right) \times \text{experim. data on } K_{\mu 2} \pi_{\mu 2} \text{ and } K_{l3}$$

* In the SM $R_{\mu 23} = 1$. Not true for some BSM theories (for example, charged Higgs)

* Current value $R_{\mu 23} = 0.999(7)$, limited by lattice inputs.

2. Strategy: semileptonic decays with HISQ quarks

Semileptonic decays at $q^2 = 0$: Extraction of CKM matrix elements

$$K \to \pi l \nu \quad \to \quad |V_{us}|$$

$$D \to \pi(K) l \nu \quad \to \quad |V_{cd(cs)}|$$

* Analysis round 1: $K \to \pi l \nu$ analysis on the $N_f = 2+1$ Asqtad bf MILC ensembles (HISQ on Asqtad calculation): Nearly finished.

$$K \to \pi l \nu \quad \to \quad |V_{us}|$$

D **semileptonic decays at** $q^2 \neq 0$: Comparing the shape with experiment

* Test lattice QCD.

* Global fit in SM + experiment $\to$ $|V_{cs(cd)}|$ and $f_+^{D\to K(\pi)}(q^2)$

3. Form factors at $q^2 = 0$

3.1. Methodology

For the extraction of $|V_{f_1 f_2}|$ we need $f_+^{P1 \to P_2}(0)$ for mesons P_1 and P_2.

$$\langle P_2 | V^\mu | P_1 \rangle = f_+^{P_1 P_2}(q^2) \left[p_{P_1}^\mu + p_{P_2}^\mu - \frac{m_{P_1}^2 - m_{P_2}^2}{q^2} q^\mu \right] + f_0^{P_1 P_2}(q^2) \frac{m_{P_1}^2 - m_{P_2}^2}{q^2} q^\mu$$

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We use HPQCD method for D semileptonic decays

* In the continuum, the Ward identity ($S = \bar{a}b$)

$$q^\mu \langle P_2|V_\mu^{cont.}|P_1\rangle = (m_b - m_a)\langle P_2|S^{cont}|P_1\rangle$$

relates matrix elements of vector and scalar currents. In the lattice

$$q^\mu \langle P_2|V_\mu^{lat.}|P_1\rangle Z = (m_b - m_a)\langle P_2|S^{lat.}|P_1\rangle$$

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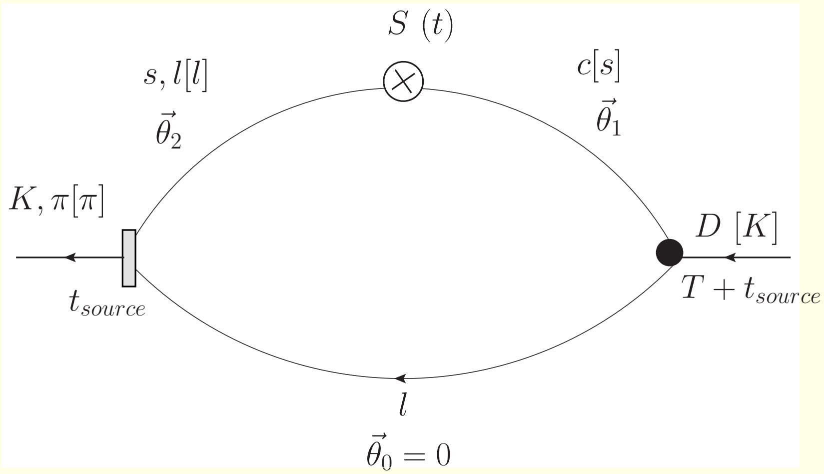
$$q^\mu\langle P_2|V_\mu^{lat.}|P_1\rangle Z = (m_b - m_a)\langle P_2|S^{lat.}|P_1\rangle$$

$\to$ replace the V_μ with an S current in the 3-point function

$$f_0^{P_1 P_2}(q^2) = \frac{m_b - m_a}{m^2_{P_1} - m^2_{P_2}}\langle P_2|S|P_1\rangle_{q^2} \Longrightarrow \boxed{f_+^{P_1 P_2}(0) = f_0^{P_1 P_2}(0) = \frac{m_b - m_a}{m^2_{P_1} - m^2_{P_2}}\langle S\rangle_{q^2=0}}$$

3.2. Simulations setup



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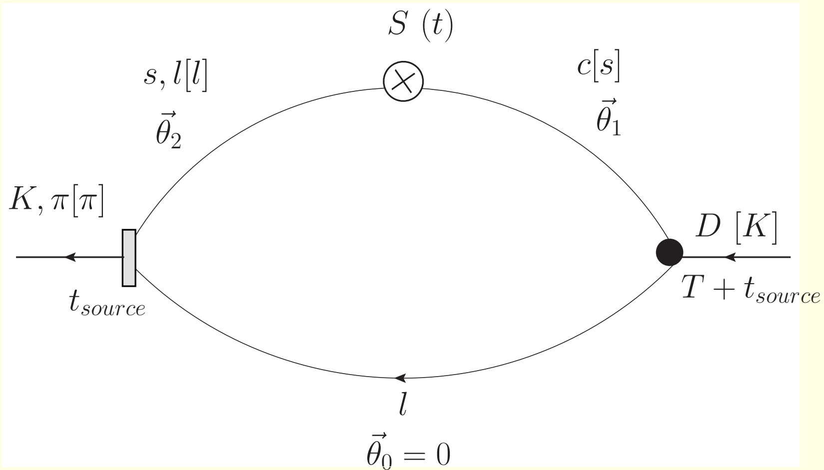

Quantities inside [ ] correspond to $K \to \pi l \nu$

* **Color random wall sources** $\to$ Reduction of statistical errors by a factor of 2-3

* **Twisted boundary conditions** $\to$ allow generating correlation functions with non-zero external momentum such that $q^2 \simeq 0$ (or any other q^2)

** $K \to \pi l \nu$: momentum injected on the K ($\vec{\theta}_1 \neq 0$) or π ($\vec{\theta}_2 \neq 0$)

Example: $q^2 = 0$ $\quad \vec{\theta}_1(q^2 = 0) = \sqrt{\left(\frac{m_K^2 + m_\pi^2}{2 m_\pi}\right)^2 - m_K^2}\,\frac{L}{\pi} \quad \Longrightarrow \quad \vec{p}_K = \vec{\theta}_1 \frac{\pi}{L}$

$$\vec{\theta}_2(q^2 = 0) = \sqrt{\left(\frac{m_K^2 + m_\pi^2}{2 m_K}\right)^2 - m_\pi^2}\,\frac{L}{\pi} \quad \Longrightarrow \quad \vec{p}_\pi = \vec{\theta}_2 \frac{\pi}{L}$$

** $D \to K(\pi) l \nu$: D-meson always at rest. Momentum injected on the $K(\pi)$ ($\vec{\theta}_0 = \vec{\theta}_1 = 0$, $\vec{\theta}_2 \neq 0$)

3.3. Analysis on the Asqtad $N_f = 2 + 1$ MILC ensembles

3.3.1 Simulation details: Lattice actions

\# Sea quarks: $N_f = 2 + 1$ **MILC** configurations with improved staggered Asqtad u, d and s sea quarks, and improved glue

RMP 82, 1349 (2010) [0903.3598] and references therein

* Asqtad: Tree-level order a^2 effects removed
  $\rightarrow$ leading errors are $\mathcal{O}(\alpha_s a^2)$, $\mathcal{O}(a^4)$

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Valence quarks: HISQ action.

E. Follana et al, HPQCD coll., Phys.Rev.D75:054502 (2007)

Highly reduce $\mathcal{O}(a^2\alpha_s)$ and $\mathcal{O}((am_Q)^4)$ errors compared to Asqtad

$\rightarrow$ more continuum-like behavior

3.3.1 Simulation details: parameters

HISQ valence quarks on $N_f = 2+1$ Asqtad MILC configurations

$\approx a$ (fm)	am_l/am_s	Volume	N_{conf}	$N_{sources}$	N_T
0.12	0.4	$20^3 \times 64$	2052	4	5
	0.2	$20^3 \times 64$	2243	4	8
	0.14	$20^3 \times 64$	2109	4	5
	0.1	$24^3 \times 64$	2098	8	5
0.09	0.4	$28^3 \times 96$	1996	4	5
	0.2	$28^3 \times 96$	1946	4	5

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* Light valence quark masses: $\frac{m_l^{val}(HISQ)}{m_s^{phys}(HISQ)} = \frac{m_l^{sea}(Asqtad)}{m_s^{phys}(Asqtad)}$

3.3.2 Fitting and statistical errors

We want to extract the value of the form factor $f_0(q^2)$ from the relation

$$f_0(q^2) = \frac{m_s - m_q}{m_K^2 - m_\pi^2} \langle S \rangle_{q^2=0} = \frac{1}{2} A_{00}(q^2) \sqrt{2E_\pi^0 2E_K^0} \, \frac{m_s - m_q}{m_K^2 - m_\pi^2}$$

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Strategy: Combined fits of two-point functions with and without external momentum (4) + three-point functions with $q^2 = 0$ (2):

$$C_{3pt}^{K\to\pi}(t, T; \vec{p}_\pi, \vec{p}_K) \quad = \quad \sum_{m,n=0}^{N_{exp}^{3pt}} (-1)^{mt} (-1)^{n(T-t)} A_{mn}(q^2) \sqrt{Z_m^{\pi,\vec{p}_\pi} Z_n^{K,\vec{p}_K}}$$

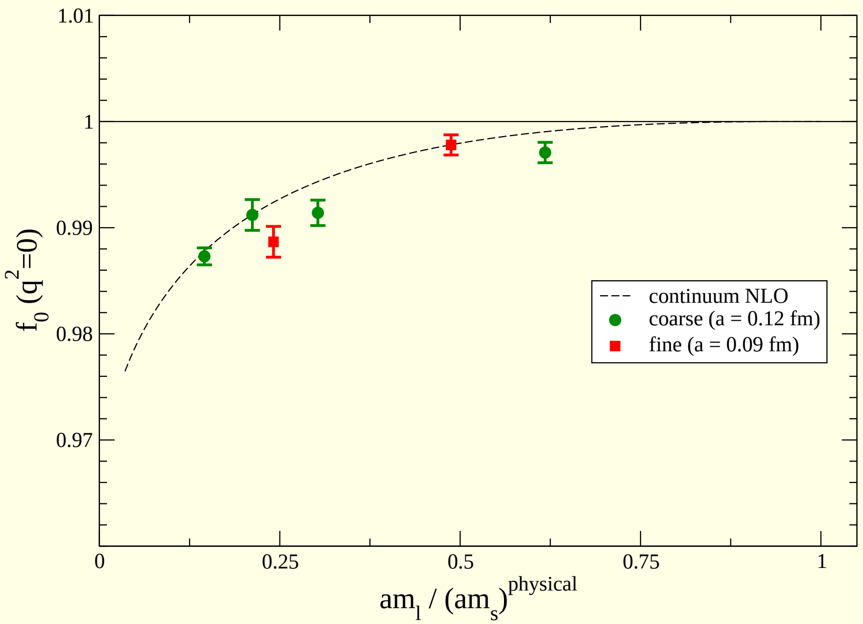
$$\times \left(e^{-E_\pi^m t - E_\pi^m (L_t - t)}\right) \left(e^{-E_K^n (T-t) - E_K^n (T - L_t + t)}\right) ;$$

$$C_{2pt}^P(t; \vec{p}_P) \quad = \quad \sum_m^{N_{exp}^{2pt}} (-1)^{mt} \sqrt{Z_m^{P,\vec{p}_P}}\, e^{-E_P^m t - E_P^m (L_t - t)} \qquad P = \pi, K$$

* Use several (3 or 4) values of T (even and odd) to fit out oscillatory terms.

3.3.2 Fitting and statistical errors

Statistical errors $0.1 - 0.15\%$.



Find it very difficult to make changes in the fitting procedure that change the fit results outside the one statistical sigma range

* Choice of source-sink separation $T's$, number of exponentials, time ranges, fitting function.

3.3.3 Chiral and continuum extrapolation

The form factor $f_+(0)$ can be written in χPT as

$$f_+(0) = 1 + f_2 + f_4 + f_6 + ... = 1 + f_2 + \Delta f$$

- # $f_+(0)$ goes to 1 in the $SU(3)$ limit due to vector current conservation
- # Ademollo-Gatto theorem $\rightarrow$ SU(3) breaking effects are second order in $(m_K^2 - m_\pi^2)$ and f_2 is completely fixed in terms of experimental quantities.

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Ademollo-Gatto theorem $\rightarrow$ SU(3) breaking effects are second order in $(m_K^2 - m_\pi^2)$ and f_2 is completely fixed in terms of experimental quantities.

* At finite lattice spacing systematic errors can enter due to violations of the dispersion relation needed to derive

$$f_+(0) = f_0(0) = \frac{m_s - m_q}{m_K^2 - m_\pi^2} \langle S \rangle_{q^2=0}$$

Dispersion relation violations in our data are $\leq 0.15\%$.

3.3.3 Chiral and continuum extrapolation

\# **Fitting strategy I**:

One-loop (NLO) partially quenched `Staggered` χPT +

two-loop (NNLO) continuum χPT by **Bijnens & Talavera**, arXiv:0303103.

$$f_+^{K\pi}(0) = 1 + f_2^{PQ\,,stag.}(a) + C_a \left(\frac{a}{r_1}\right)^2 + f_4^{cont.}(\mathrm{logs}) + f_4^{cont.}(L_i' s)$$
$$+ r_1^4 (m_\pi^2 - m_K^2)^2 \left[C_6'^{(1)} + C_6^a \left(\frac{a}{r_1}\right)^2 \right]$$

where $C_6'^{(1)} \propto C_{12} + C_{34} - L_5^2$.

L_5 is an $\mathcal{O}(p^4)$ LEC and $C_{12,34}$ are $\mathcal{O}(p^6)$ LECs

* `Staggered` χPT: logs are known non-analytical functions of $m_{K,\pi}$ containing dominant taste-breaking a^2 effects
$\to$ remove the dominant light discretization errors

3.3.3 Chiral and continuum extrapolation

Fitting strategy II:

One-loop (NLO) partially quenched `Staggered` χPT +

analytical parametrization of NNLO terms.

$$
\begin{aligned}
f_+^{K\pi}(0) = 1 + f_2^{PQ\,,stagg.}(a) + C_a\left(\frac{a}{r_1}\right)^2 + r_1^4\,(m_\pi^2 - m_K^2)^2 \\
\times\left[C_6^{(1)}(r_1 m_\pi)^2 + C_6^{(2)}(r_1 m_K)^2 + C_6^a\left(\frac{a}{r_1}\right)^2\right]
\end{aligned}
$$

* We also add terms of order $(r_1 m_\pi)^4$, $(r_1 m_\pi)^2 log((r_1 m_\pi)^2)$.

3.3.3 Chiral and continuum extrapolation

Results: some examples

Estimate errors using 500 bootstrap ensembles.

SχPT expressions used are not complete.

* Not all hairpin terms are included in the fitting function (need to be checked).

* Mixed-action pion mass splittings are approximated by

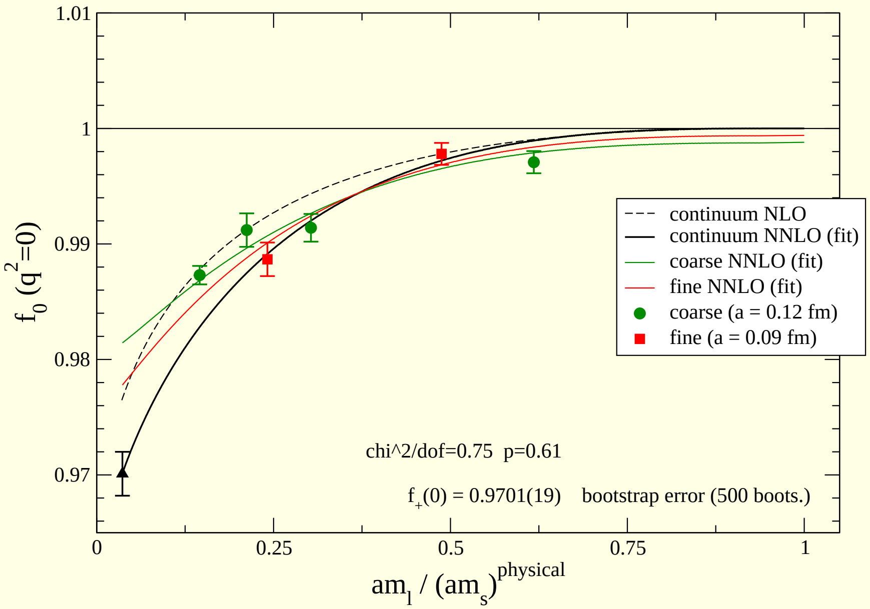
$$\Delta_{mix} = (\Delta_{sea}(Asqtad) + \Delta_{valence}(HISQ))/2$$

** Using the correct splittings does not change the central values by more than 0.1%.

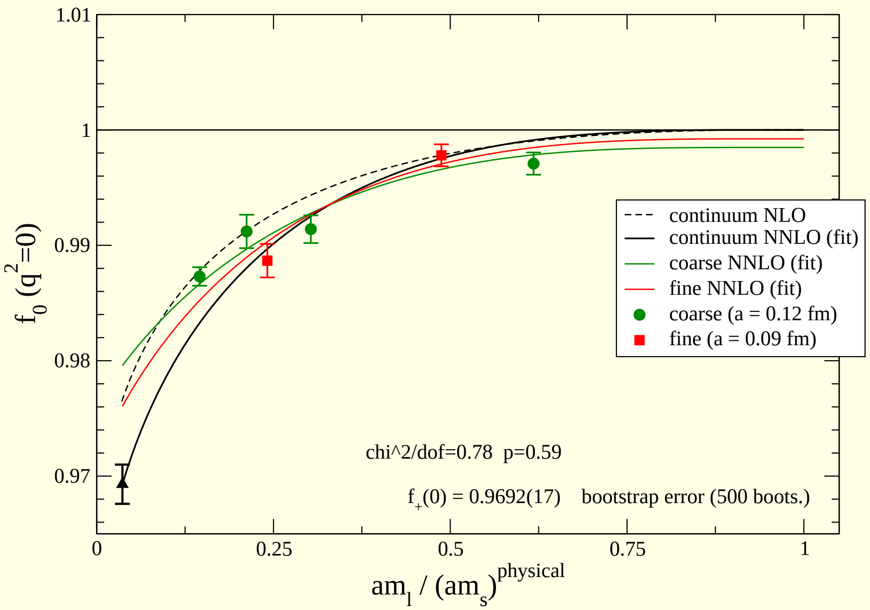
3.3.3 Chiral and continuum extrapolation

Results: some examples

Example **fitting strategy I**



Example **fitting strategy II**



Priors central values for $L_i' s$ from **Bijnens and Jemos**, 1103.5945, widths $10\times$ larger than the erros quoted there

$$1 + f_2^{PQ,stagg.}(a) + C_a \left(\frac{a}{r_1}\right)^2 + (m_\pi^2 - m_K^2)^2 \left[C_6^{(1)}(r_1 m_\pi)^2 + C_6^{(2)}(r_1 m_K)^2\right]$$

Different choices of fitting function tested within strategies **I** and **II**.

3.3.3 Chiral and continuum extrapolation

Main features of the fits.

* Different results (fitting functions, fitting strategies, ...) agree within one statistical σ.

* Statistical (bootstrap) errors around $0.2 - 0.3\%$.

* Violations of AG theorem are $\sim 0.32 - 0.15\%$ for $a \approx 0.12$ fm and $\sim 0.15 - 0.1\%$ for $a \approx 0.09$ fm.

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Final strategy for chiral and continuum extrapolation not decided yet.

Need to check $\mathtt{S}\chi\mathtt{PT}$ and clarify the origin of a^2 effects.

3.3.4 Expected error budget

* Statistical+extrapolation: 0.2-0.3%
* Chiral extrapolation/fitting function: 0.1%
* Discretization errors: 0.15-0.2%
  ** Spread of results when adding a^2, $a^2\alpha_s$, a^4, $(m_K - m_\pi^2)^2 a^2$ $(m_K - m_\pi^2)^2 a^2$, $E_P^2 a^2$ and/or $p_P^2 a^2$ terms in the fitting function
  ** Deviation from continuum dispersion relation $\leq 0.15\%$
* Mistuning of m_s on the sea: 0.2%
* Finite volume effects: ?
  Explicit check on a larger volume ($a = 0.12\ fm$, $am_l = 0.2am_s$, $V = 28^3 \times 64$)

TOTAL: 0.35-0.5%

RBC/UKQCD, EPJC69(2010): $f_+(0) = 0.9599(34)(^{+31}_{-43})(14)$ $(0.5 - 0.6\%)$

ETMC, PRD80(2009): $f_+(0) = 0.9560(84)$ (0.9%)

3.4. HISQ valence quarks on HISQ $N_f = 2 + 1 + 1$ MILC ensembles

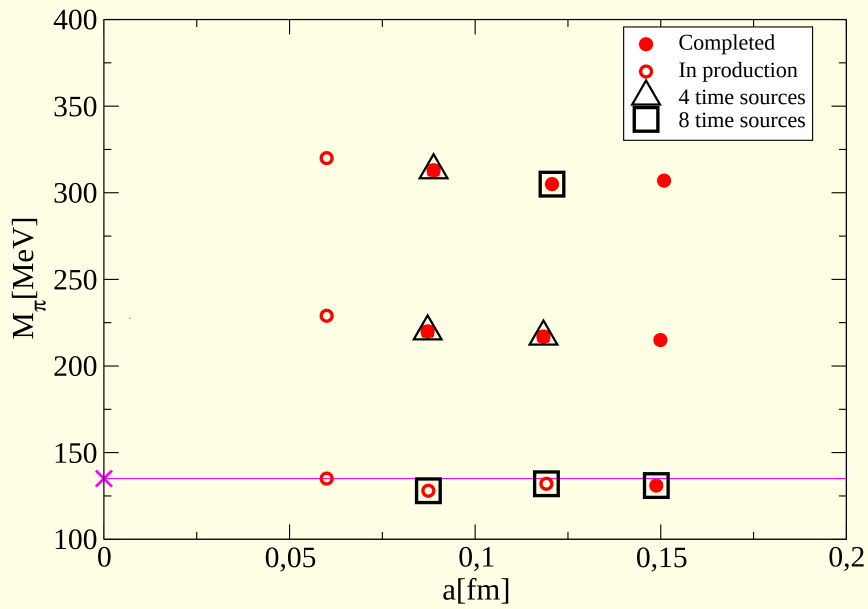
3.4.1 Simulation details

\# Same set-up as for the Asqtad on HISQ calculation.

\# Data generated for $K \to \pi l \nu$ and $D \to K(\pi) l \nu$ at $q^2 = 0$ (and $q^2 = q^2_{max}$).

Planned runs

* ~ 1000 configurations per ensemble.
* 4 or 8 time sources.
* $4 - 5$ source-sink separations.



$$am_l^{valence} = am_l^{sea},\ am_s^{valence} = am_s^{physical},\ \text{and}$$
$$am_c^{valence} = am_c^{sea}, \approx am_c^{phys}.$$

3.4.2 First preliminary results

Improvements

* Reduction of discretization errors from the sea .
* Physical quark masses.
* Incorporates effects of m_c^{sea}.
* Better tuning of sea quark masses (especially am_s).

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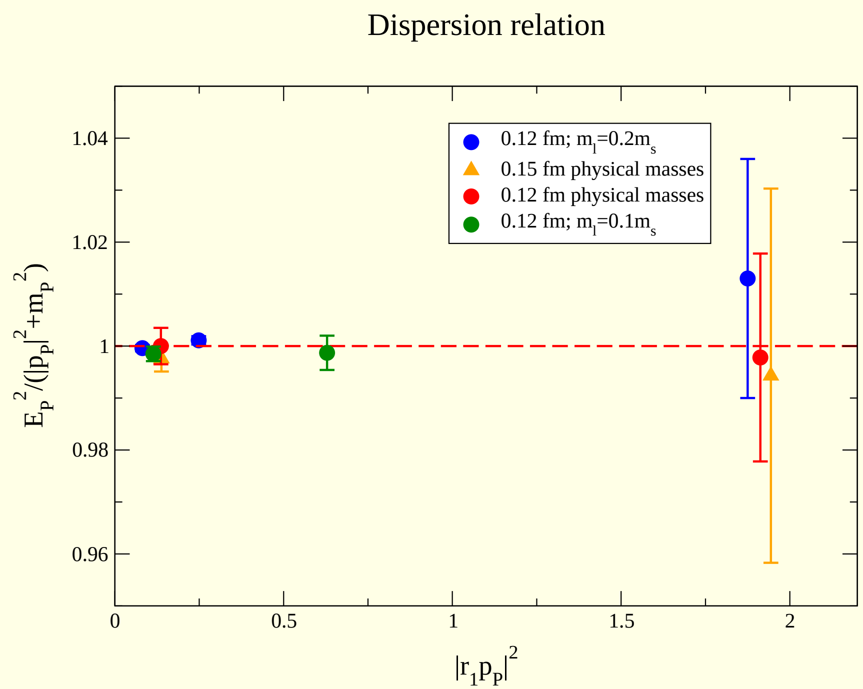

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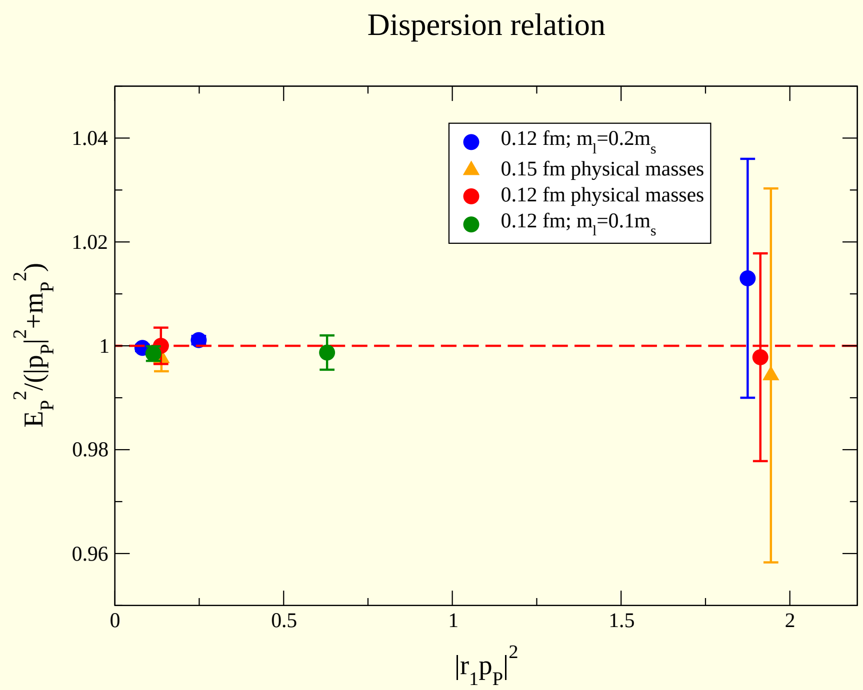


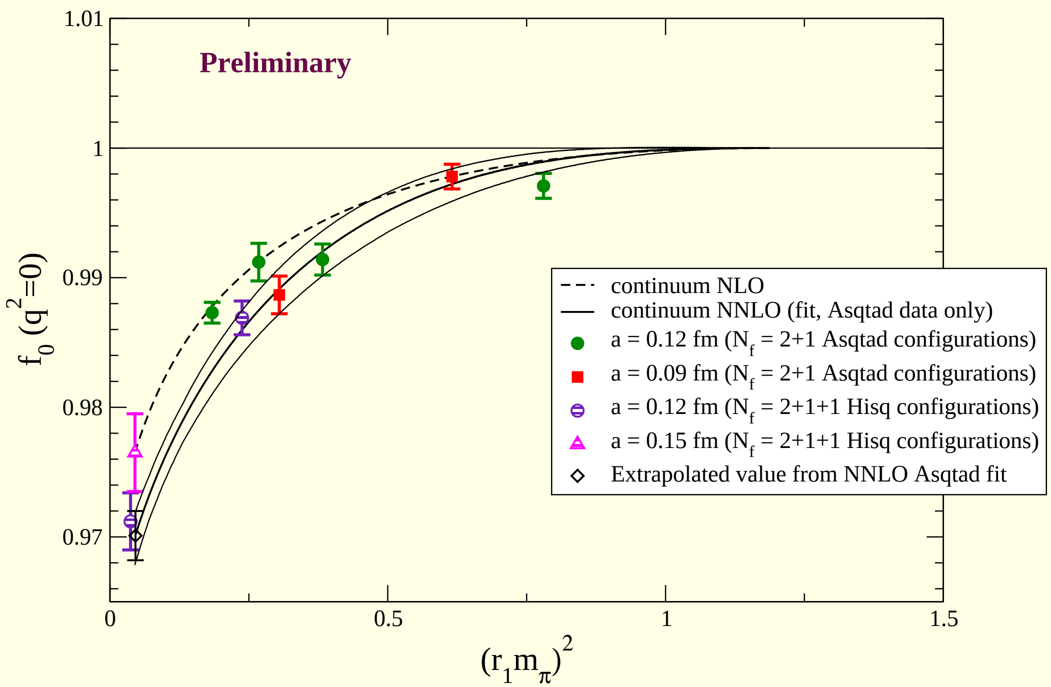
* Statistical errors are larger for smaller quark masses and the external momentum ap needed for $q^2 = 0$ is larger.

$\rightarrow$ Need data at unphysical light masses to reduce statistical and discretization errors

Semileptonic decays at $q^2 = 0$: $N_f = 2+1+1$

First very preliminary results

$K \to \pi l \nu$



Reduced discretization errors

\# $D \to K(\pi) l \nu$ Still working on the (more challenging) fits

4. Conclusions

Calculation of $f_+^{K\pi}(q^2=0)$ with HISQ valence quarks on the $N_f=2+1$ Asqtad MILC configurations nearly complete.

* We expect errors $\sim 0.35-0.5\%$.

* Dominant sources of uncertainty are chiral extrapolation, discretization effects, and mistuning of m_s on the sea.

We have started a broader calculation of $K\to\pi l\nu$ and $D\to K(\pi)l\nu$ form factors at $q^2=0$ on the $N_f=2+1+1$ HISQ MILC config.

* Include physical quark mass results $\to$ reduction of chiral extrapolation uncertainty.

* Better tuning of sea quark masses $\to$ reduction of m_s uncertainty.

* Preliminary results for $f_+^{K\pi}(0)$ indicates small discretization effects.

$f_+^{DK(\pi)}(0)$ in progress.

$f_+^{DK}(q^2)$ and $f_+^{D\pi}(q^2)$ with $q^2\neq 0$ in the near future.

2.4.1 χPT: analytic NNLO

χ^2/dof	p	C_a	$C_6^{(1)}$	$C_6^{(2)}$	C_6^a	$f_+(0)$
0.78	0.59	−0.011(11)	0.033(27)	−0.024(6)	0	0.9692(17)
0.91	0.48	0	0.014(21)	−0.024(11)	−0.03(14)	0.9689(28)
0.67	0.67	−0.022(19)	0.065(48)	−0.033(13)	0.19(23)	0.9669(31)
priors		0 ± 1	$0 \pm s^2$	$0 \pm s^2$	$0 \pm s^2$	

where $s = 1/(8\pi^2(f_\pi r_1)^2 \simeq 0.6$.

$$f_+^{K\pi}(0) = 1 + f_2^{PQ\,,stagg.}(a) + C_a \left(\frac{a}{r_1}\right)^2 + r_1^4\,(m_\pi^2 - m_K^2)^2 \times \left[C_6^{(1)}(r_1 m_\pi)^2 + C_6^{(2)}(r_1 m_K)^2 + C_6^a \left(\frac{a}{r_1}\right)^2 \right]$$

2.4.2. χPT: continuum NNLO

Fit	C_a	C_6^a	χ^2/dof	p	$f_+(0)$	$(C_{12}+C_{34})\times 10^6$
I	-0.015(8)	0	0.91	0.48	0.9692(17)	3.9(3)
II	-0.015(8)	0	0.9	0.5	0.9693(17)	3.9(3)
III	-0.009(11)	0	0.75	0.61	0.9701(19)	4.0(4)
IV	-0.007(11)	0	0.76	0.6	0.9699(19)	5.3(4)
III	0	-0.02(14)	0.86	0.53	0.9700(33)	4.4(4)
I	-0.017(8)	0.15(13)	0.7	0.65	0.9671(24)	4.3(4)
III	-0.016(16)	0.13(21)	0.69	0.66	0.9677(33)	4.0(4)

$$f_+^{K\pi}(0) = 1 + f_2^{PQ,stag.}(a) + C_a\left(\frac{a}{r_1}\right)^2 + f_4^{cont.}(\text{logs}) + f_4^{cont.}(L_i's)$$
$$+r_1^4(m_\pi^2 - m_K^2)^2\left[C_6'^{(1)} + C_6^a\left(\frac{a}{r_1}\right)^2\right]$$

where $C_6'^{(1)} \propto C_{12} + C_{34} - L_5^2$.

2.4.2. χPT: continuum NNLO

I. Fix $L_i's$ to Bijnens' values.

II. Free $L_i's$ with priors and widths equal to Bijnens' values.

III. Free $L_i's$ with priors equal to Bijnens' values and widths $10\times$ larger.

IV. Free $L_i's$, same as III for L_{1-3} and use MILC determination in PoS LAT2009:079(2009) for the prior/width(twice de error) of $L_{4,5}$.