Coupled Two-Particle Channels in Finite Volume

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Outline

- Introduction
- Generalized Lüscher method
- Generalized Lellouch-Lüscher formula
- Summary

Introduction and Motivation

The finite volume effects for kaon decay were worked out by Martin Lüscher and Laurent Lellouch over a decade ago.[1][2][3][4]

More recently, lattice simulations have made significant progress towards a first principle determination of these weak decay amplitudes.[5][6]

[1] Luescher, M. *Commun. Math. Phys.* **104,** 177 (1986).
[2] Luescher, M. *Commun. Math. Phys.* **105,** 153–188 (1986).
[3] Luescher, M. *Nucl. Phys.* **B354,** 531–578 (1991).
[4] Lellouch, L. & Luscher, M. *Commun.Math.Phys.* **219,** 31–44 (2001).
[5] Blum, T. *et al.* *Phys.Rev.* **D84,** 114503 (2011).
[6] Blum, T. *et al.* arXiv:1111.1699 [hep-lat] (2011).

Introduction and Motivation

What about heavier mesons?
For example, LHCb recently reported CP-violation in the difference of CP-asymmetries of[7]

$$D^0 \to \pi^- \pi^+ \quad \text{and} \quad D^0 \to K^- K^+$$

Is this consistent with the Standard Model?

[7] Aaij, R. *et al.* *Phys. Rev. Lett.* **108**, 111602 (2012).

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Is this consistent with the Standard Model?

Finite volume effects for these decays are not yet worked out. I present here a first step towards controlling these effects.

[7] Aaij, R. *et al.* *Phys. Rev. Lett.* **108,** 111602 (2012).

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Set-up

Here finite volume means...

- finite, cubic spatial volume (extent L)
- periodic boundary conditions
- time direction infinite.

Assume L large enough to ignore exponentially suppressed corrections.

Assume continuum field theory throughout.

Allow non-zero total momentum in finite volume frame...

- total energy E
- total momentum $\vec{P}$ $\quad \left(\vec{P} = \frac{2\pi \vec{n}_P}{L} \qquad \vec{n}_P \in \mathbb{Z}^3\right)$
- CM frame energy E^* $\quad \left(E^* = \sqrt{E^2 - \vec{P}^2}\,\right)$

Set-up

Arbitrary number (N) of open channels...

- Restrict, however, to **two-particle channels** with **scalar particles**.
- Allow identical or non-identical particles.
- Allow non-degenerate masses.

For example $N = 3$ with $\pi\pi$, $K\overline{K}$ and $\eta\eta$.

Define q_i^* as the magnitude of CM frame momentum for the ith channel.

Statement of the problem

Infinite volume theory described by $2 \to 2$ scattering amplitudes

$$i\mathcal{M}_{jk}$$

$$\text{where} \quad \begin{array}{l} k = 1, \cdots, N \text{ is the in-state} \\ j = 1, \cdots, N \text{ is the out-state.} \end{array}$$

Want to relate $\mathcal{M}$ to the discrete spectrum of the finite volume theory

$$E_k^* \qquad \text{for} \qquad k = 1, 2, 3, \cdots$$

at a given $\{L, \vec{n}_P\}$.

Found the result by generalizing work of Kim, Sachrajda and Sharpe.[8]

[8] Kim, C. *et al.* *Nucl.Phys.* **B727,** 218–243 (2005).

Definition of F

To state the result define $F_{ij;\ell_1,m_1;\ell_2,m_2}$

$$F_{ij;\ell_1,m_1;\ell_2,m_2} \equiv \delta_{ij}\eta_i \left[\frac{\mathrm{Re} q_i^*}{8\pi E^*} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} + \right.$$
$$\left. \frac{i}{2\pi E L} \sum_{\ell,m} x_i^{-\ell} \, \mathcal{Z}^P_{\ell m}[1; x_i^2] \int d\Omega \, Y^*_{\ell_1,m_1} Y^*_{\ell,m} Y_{\ell_2,m_2} \right]$$

where $x_i \equiv q_i^* L/(2\pi)$ and $\mathcal{Z}^P_{\ell m}$ is a generalization of the zeta-function.

$\eta = 1/2$ for identical particles and 1 for non-identical.

Observe that F is...

- diagonal in channel space
- not diagonal in angular momentum space (rotational symmetry broken).

Definition of $\mathcal{M}$

We also define the partial wave scattering amplitude $\mathcal{M}_{ij;\ell_1,m_1;\ell_2,m_2}$

$$\mathcal{M}_{ij}(\hat{k}^*, \hat{k}'*) \equiv 4\pi \mathcal{M}_{ij;\ell_1,m_1;\ell_2,m_2} Y_{\ell_1,m_1}(\hat{k}^*) Y^*_{\ell_2,m_2}(\hat{k}'^*)$$

In contrast to F, $\mathcal{M}$ is...

- not diagonal in channel space
- diagonal in angular momentum space.

$$\mathcal{M}_{ij;\ell_1,m_1;\ell_2,m_2} = \mathcal{M}_{ij}^{\ell_1,m_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$$

Generalized Lüscher method

We find that the spectrum

$$E_k^* \qquad \text{for} \qquad k = 1, 2, 3, \cdots$$

at a particular $\{L, \vec{n}_P\}$ is given by solutions to

$$\det(F^{-1} + i\mathcal{M}) = 0\,.$$

Agrees with work by Briceno and Davoudi (submitted simultaneously).[9]

Earlier work found equivalent result for limiting case of only s-wave scattering and $\vec{P} = 0$.[10]

[9] Briceno, R. A. & Davoudi, Z. arXiv:1204.1110 [hep-lat] (2012).
[10] Bernard, V. *et al.* *JHEP* **1101**, 019 (2011).

Generalized Lüscher method

$$\det(F^{-1} + i\mathcal{M}) = 0\,.$$

Result holds for

- any number of strongly coupled channels (but only two-particle, scalar channels)
- any total momentum $\vec{P}$
- arbitrary values of the different partial wave amplitudes $\mathcal{M}_{ij;\ell_1,m_1;\ell_2,m_2}$

F is not diagonal in angular momentum. However if

$$\mathcal{M}_{ij}^{\ell>\ell_{max},m} = 0$$

then one can replace F with finite version.

Two channel, s-wave limit

From now on assume

- $\ell_{max} = 0$ (only keep s-wave)
- N=2 (only two open channels)

In particular consider toy model of pions and kaons with $M_\pi < M_K < 2M_\pi$.

Assume that $2M_K < E^* < 4M_\pi$, so the only open channels are

$$1: \ \pi\pi \ \ \text{(identical)} \qquad \text{and} \qquad 2: \ K\overline{K} \ \ \text{(nonidentical)}\,.$$

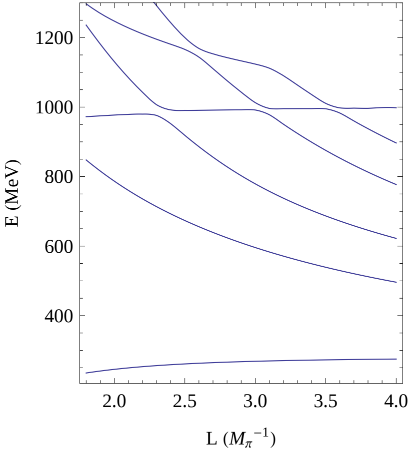
We find

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P}) \equiv \det\left[\begin{pmatrix} (F_1^s)^{-1} & 0 \\ 0 & (F_2^s)^{-1} \end{pmatrix} + i \begin{pmatrix} \mathcal{M}^s_{1\to1} & \mathcal{M}^s_{2\to1} \\ \mathcal{M}^s_{1\to2} & \mathcal{M}^s_{2\to2} \end{pmatrix}\right] = 0$$

Curves from UChPT

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P} = 0) = 0$$

Curves predicted by unitarized chiral perturbation theory (UChPT)



S-matrix

Next note $$i\left(\ \mathcal{M}\ \right) = \left(\ Q\ \right)\left[\left(\ S\ \right) - 1\right]\left(\ Q\ \right)$$

where $\left(\ Q\ \right) = \sqrt{4\pi E^*}\begin{bmatrix}(q_1^*\eta_1)^{-1/2} & 0 \\ 0 & (q_2^*\eta_2)^{-1/2}\end{bmatrix}$,

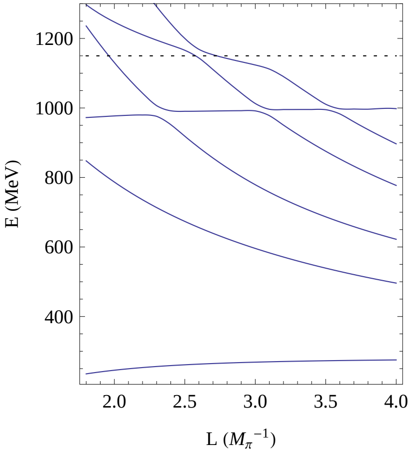
and $\left(\ S\ \right)$ is a dimensionless, 2×2, symmetric, unitary matrix.

Determined by 3 real parameters

Extraction of amplitude

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P} = 0) = 0$$

Extraction of the scattering amplitude at a given energy E^*



Extraction of amplitude

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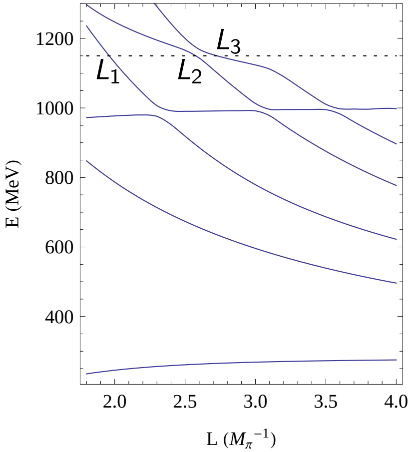
Extraction of the scattering amplitude at a given energy E^*



$$\Delta^{\mathcal{M}}(L_1, E^*, \vec{P}_1) = 0$$
$$\Delta^{\mathcal{M}}(L_2, E^*, \vec{P}_2) = 0$$
$$\Delta^{\mathcal{M}}(L_3, E^*, \vec{P}_3) = 0$$

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Add in another particle

So far we have given the relation between $\mathcal{M}$ and E_k^* for a toy theory with two channels

$$1: \ \pi\pi \ \ (\text{identical}) \qquad \text{and} \qquad 2: \ K\overline{K} \ \ (\text{nonidentical}),$$

Now consider an additional scalar particle in the toy model.

Call it the D-meson and take

$$2M_\pi < 2M_K < M_D < 4M_\pi \,.$$

This is a bad approximation for the actual D-meson.

Statement of Problem

Define $\mathcal{H}_W$ as a weak hamiltonian density which couples D to $\pi\pi$ and $K\overline{K}$.

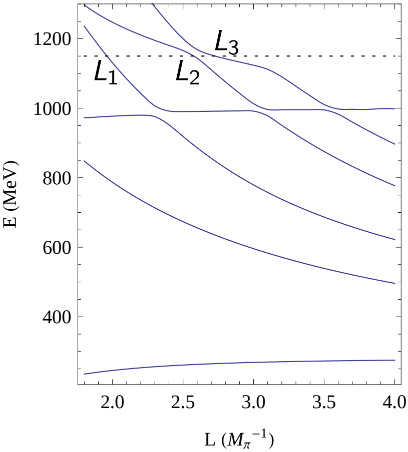
We want a method to determine infinite volume weak decay elements

$$A_{D\to\pi\pi} \equiv \langle \pi\pi | \mathcal{H}_W | D \rangle \quad \text{and} \quad A_{D\to KK} \equiv \langle K\overline{K} | \mathcal{H}_W | D \rangle \,,$$

from finite volume quantities.

Tuning degeneracy

Identify $\{L, \vec{n}_P\}$ that put M_D in the spectrum.



$$\Delta^{\mathcal{M}}(L_1, M_D, \vec{P}_1) = 0$$
$$\Delta^{\mathcal{M}}(L_2, M_D, \vec{P}_2) = 0$$
$$\Delta^{\mathcal{M}}(L_3, M_D, \vec{P}_3) = 0$$

$$M_{D\to n} = \langle n|\mathcal{H}_W(0)|D\rangle$$

Generalized Lellouch-Lüscher formula

We want a formula that takes as input $\mathcal{M}$ and $M_{D\to n}$ and gives from this

$$A_{D\to\pi\pi} \equiv \langle \pi\pi|\mathcal{H}_W|D\rangle \quad \text{and} \quad A_{D\to KK} \equiv \langle K\overline{K}|\mathcal{H}_W|D\rangle \,.$$

To derive this incorporate the weak interaction into the hamiltonian

$$\mathcal{H}(x) \longrightarrow \mathcal{H}(x) + \lambda\mathcal{H}_W(x)\,,$$

where λ is a real free parameter.

Generalized Lellouch-Lüscher formula

The new hamiltonian

$$\mathcal{H}(x) + \lambda \mathcal{H}_W(x)$$

changes the secular equation to

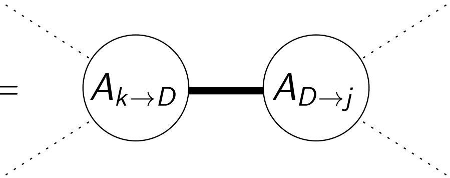
$$\Delta^{\mathcal{M}+\Delta\mathcal{M}}(L, E^*, \vec{P}) = 0 \, .$$

Generalized Lellouch-Lüscher formula

The change $\Delta\mathcal{M}$ is related to the infinite volume decay amplitudes through terms proportional to

$$A_{\pi\pi\to D}A_{D\to\pi\pi} \qquad A_{KK\to D}A_{D\to\pi\pi}$$
$$A_{\pi\pi\to D}A_{D\to KK} \qquad A_{KK\to D}A_{D\to KK}$$

$$i\Delta\mathcal{M}_{jk} =$$



Generalized Lellouch-Lüscher formula

In addition if $E^* = M_D$ is in the spectrum of the unperturbed hamiltonian, then

$$E^* = M_D \pm \lambda V |M_{D \to n}|$$

is the shifted level in the spectrum of

$$\mathcal{H}(x) + \lambda \mathcal{H}_W(x) \,.$$

Generalized Lellouch-Lüscher formula

We generate three new equations

$$
\begin{array}{ccc}
\Delta^{\mathcal{M}}(L_1, M_D, \vec{P}_1) = 0 & & \Delta^{\mathcal{M}+\Delta\mathcal{M}}(L_1, M_D \pm \lambda\Delta E^*, \vec{P}_1) = 0 \\
\Delta^{\mathcal{M}}(L_2, M_D, \vec{P}_2) = 0 & \longrightarrow & \Delta^{\mathcal{M}+\Delta\mathcal{M}}(L_2, M_D \pm \lambda\Delta E^*, \vec{P}_2) = 0 \\
\Delta^{\mathcal{M}}(L_3, M_D, \vec{P}_3) = 0 & & \Delta^{\mathcal{M}+\Delta\mathcal{M}}(L_3, M_D \pm \lambda\Delta E^*, \vec{P}_3) = 0
\end{array}
$$

Expand the new equations...at $\mathcal{O}(\lambda)$

- Terms with $A_{j\to D}A_{D\to k}$ enter through $\Delta\mathcal{M}$
- Terms with $|M_{D\to n}|$ enter through ΔE^*

Result

The final result may be written as

$$\left| C_\pi(L_n, \vec{P}_n)\ A_{D\to\pi\pi} + C_K(L_n, \vec{P}_n)\ A_{D\to KK} \right| = \left| M_{D\to n} \right| .$$

Result

The final result may be written as

$$\left|C_\pi(L_n,\vec{P}_n)\ A_{D\to\pi\pi}+C_K(L_n,\vec{P}_n)\ A_{D\to KK}\right|=\left|M_{D\to n}\right|.$$

$A_{D\to\pi\pi}$ and $A_{D\to KK}$ are complex. However one can show that, given $\mathcal{M}(E^*=M_D)$, there are actually only **two real degrees of freedom** (Watson's theorem).

The coefficients C_π and C_K depend on M_D, on $\{L_k,\vec{P}_k\}$ and also on

$$\mathcal{M}(E^*=M_D)\quad\text{and}\quad\left.\frac{d\mathcal{M}(E^*)}{dE^*}\right|_{M_D}.$$

So we have two real unknowns, and three independent equations.

Outline

Summary

Generalized Lüscher method to include an arbitrary number of two particle channels in a moving frame.

Generalized Lellouch-Lüscher formula to give decay into strongly coupled two particle channels.

Next step is four particle channels.

Relation on States

The final result may be written as

$$\left| C_\pi(L_n, \vec{P}_n)\ A_{D\to\pi\pi} + C_K(L_n, \vec{P}_n)\ A_{D\to KK} \right| = \left| M_{D\to n} \right| .$$

$$\left| C_\pi(L_n, \vec{P}_n)\ \langle \pi\pi | \mathcal{H}_W | D \rangle + C_K(L_n, \vec{P}_n)\ \langle K\overline{K} | \mathcal{H}_W | D \rangle \right| = \left| {}_L\langle n | \mathcal{H}_W | D \rangle \right| .$$

Note that this result is really a relation on states

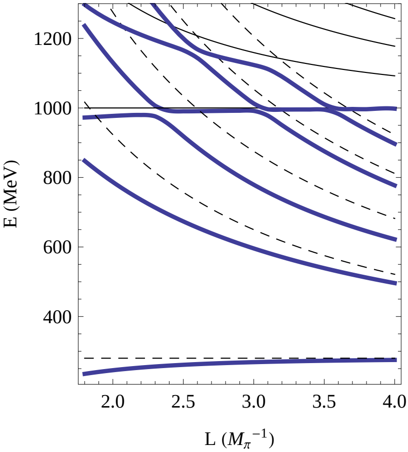
$${}_L\langle n| = C_\pi \langle \pi\pi, \mathrm{out}| + C_K \langle K\overline{K}, \mathrm{out}| + \cdots$$

where the $\cdots$ indicate higher angular momentum states which don't contribute to matrix elements.

Free curves

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P} = 0) = 0$$

Curves predicted by UChPT together with free particle curves



Parametrization

Choosing

$$\left(\; S \; \right) = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_\alpha} & 0 \\ 0 & e^{2i\delta_\beta} \end{pmatrix} \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix} ,$$

allows us to rewrite

$$\det[F^{-1} + i\mathcal{M}] = 0$$

as

$$[\tan\delta_\alpha + \tan\phi^P(q_1^*)][\tan\delta_\beta + \tan\phi^P(q_2^*)] \\ + \sin^2\epsilon \; [\tan\delta_\alpha - \tan\delta_\beta][\tan\phi^P(q_1^*) - \tan\phi^P(q_2^*)] = 0.$$

where

$$\tan\phi^P(q^*) = -\pi^{3/2}(E/E^*) \; x \; [Z_{00}^P(1; x^2)]^{-1} \, .$$

Parametrization

$$[\tan\delta_\alpha + \tan\phi^P(q_1^*)][\tan\delta_\beta + \tan\phi^P(q_2^*)] \\ + \sin^2\epsilon\ [\tan\delta_\alpha - \tan\delta_\beta][\tan\phi^P(q_1^*) - \tan\phi^P(q_2^*)] = 0$$

Observe that this reduces to the uncoupled case for

- $\epsilon = 0$
- $\tan\phi^P(q_1^*) = \tan\phi^P(q_2^*)$ (degenerate masses)
- $\tan\delta_\alpha = \tan\delta_\beta$

Alternative definition of F

$$F_{ij;\ell_1,m_1;\ell_2,m_2} \equiv \delta_{ij} F_{i;\ell_1,m_1;\ell_2,m_2}$$

$$\equiv \delta_{ij}\eta_i \left[\frac{\mathrm{Re} q_i^*}{8\pi E^*} \delta_{\ell_1\ell_2}\delta_{m_1 m_2} - \frac{i}{2E^*} \sum_{\ell,m} \frac{\sqrt{4\pi}}{q_i^{*\,\ell}} c^P_{\ell m}(q_i^{*\,2}) \int d\Omega\, Y^*_{\ell_1,m_1} Y^*_{\ell,m} Y_{\ell_2,m_2} \right]$$

$$c^P_{\ell m}(q^{*\,2}) = \frac{1}{L^3} \sum_{\vec{k}} \frac{\omega_k^*}{\omega_k} \frac{e^{\alpha(q^{*\,2}-k^{*\,2})}}{q^{*\,2}-k^{*\,2}} k^{*\,\ell} Y_{\ell,m}(\hat{k}^*)$$

$$- \delta_{\ell 0} \mathcal{P} \int \frac{d^3k^*}{(2\pi)^3} \frac{\omega_k^*}{\omega_k} \frac{e^{\alpha(q^{*\,2}-k^{*\,2})}}{q^{*\,2}-k^{*\,2}}$$

Derivation

For a given $\{L, \vec{P}\}$, the two-particle energies of the finite volume theory are the values of E where

$$C_L(E, \vec{P}) \equiv \int_L d^4x \; e^{-i\vec{P}\cdot\vec{x}+iEt} \langle \Omega | T\sigma(x)\sigma^\dagger(0)|\Omega\rangle_L$$

diverges. Here $\sigma(x)$ is an operator which couples to two particle states.

We now turn to a **nonperturbative calculation of the finite volume corrections to C_L**. We work in scalar field theory and allow all terms with even powers of the single particle interpolating fields.

Let's first consider $E^* < 2M_K$ (also $< 4M_\pi$) so that we only have to worry about two pion states.[11]

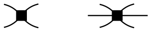
[11] Kim, C. *et al.* *Nucl.Phys.* **B727,** 218–243 (2005).

Derivation

Then $C_L(E, \vec{P})$ is equal to a sum of all Feynman diagrams built from...

- endcaps $\sigma(q)$ and $\sigma^\dagger(q')$. These are regular functions of momentum, determined by the specific form of the operators.

- arbitrary even vertices



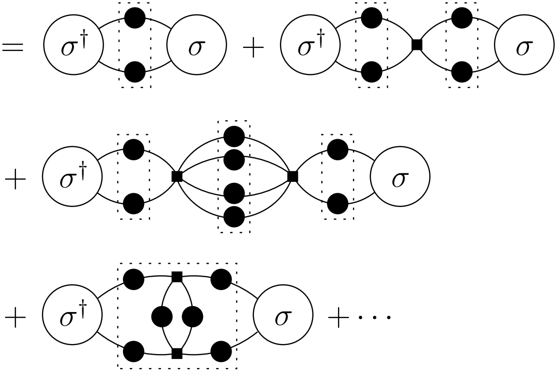
- exact pion propagators

$$= i \frac{z(q)}{(q^0)^2 - \vec{q}^{\,2} - M_\pi^2 + i\epsilon}$$

Derivation

Schematically

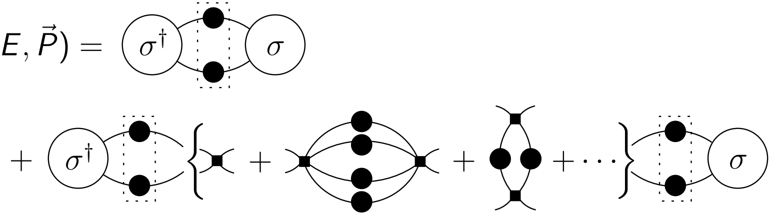
$C_L(E, \vec{P}) =$



Derivation

For the values of E^* being considered, only two propagators can go on shell.

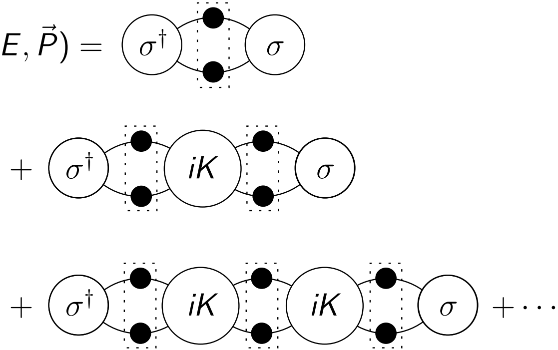
$C_L(E, \vec{P}) =$



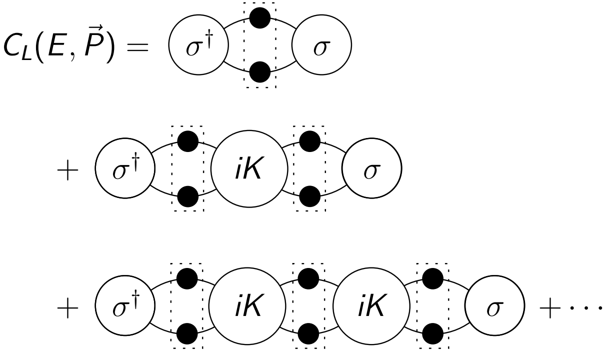
$+ \cdots$

Derivation

$C_L(E, \vec{P}) =$



Derivation



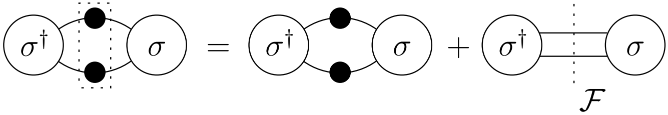
Let's focus on the first term.

Finite volume effects in first term

Implies we can write

$$\text{first term} = \text{infinite volume version of first term} + \int d\Omega d\Omega' \; \sigma^*(\hat{q})\mathcal{F}(\hat{q},\hat{q}')\sigma^{*\dagger}(\hat{q}')$$

$$\text{first term} = \text{infinite volume version of first term} - \sigma_{\ell m} F_{\ell m,\ell' m'} \sigma^\dagger_{\ell' m'}$$

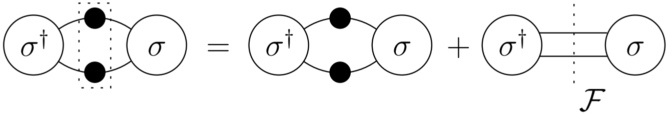


Finite volume effects in first term

Implies we can write

$$\text{first term} = \text{infinite volume version of first term} + \int d\Omega d\Omega' \, \sigma^*(\hat{q}) \mathcal{F}(\hat{q}, \hat{q}') \sigma^{*\dagger}(\hat{q}')$$

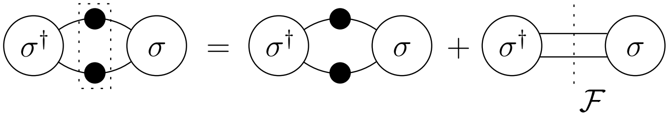
$$\text{first term} = \text{infinite volume version of first term} - \sigma_{\ell m} F_{\ell m, \ell' m'} \sigma^\dagger_{\ell' m'}$$



In the term with F only the on-shell values of the σs are needed.

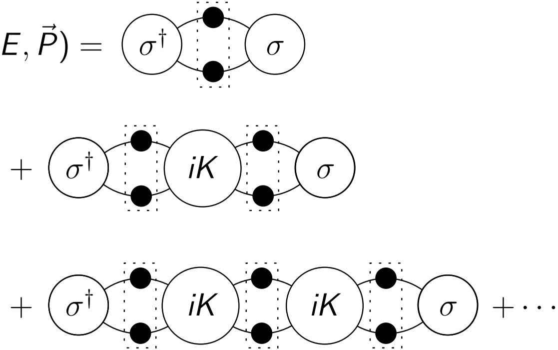
Derivation

Substitute



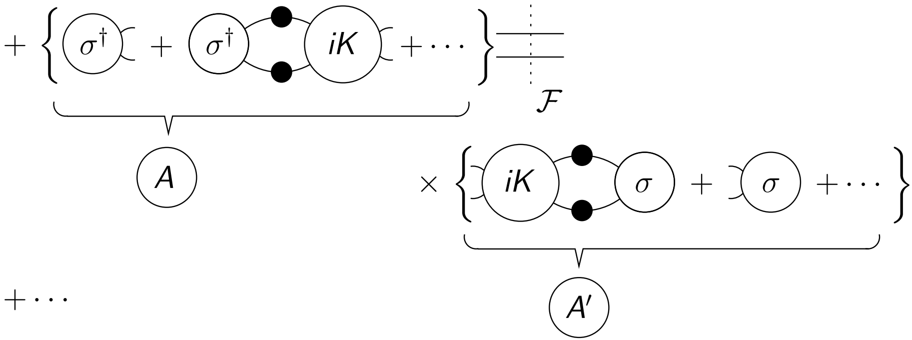
into

$C_L(E, \vec{P}) =$

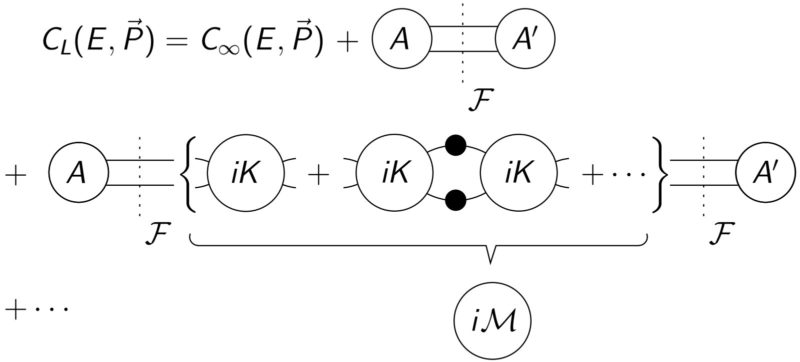


Derivation

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

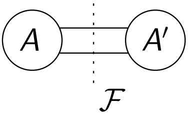


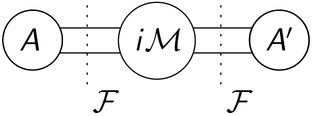
Derivation

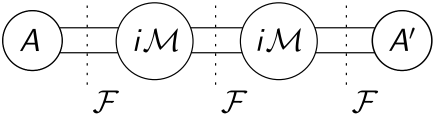


$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$
A
A'
$\mathcal{F}$
+
A
$\mathcal{F}$
iK
+
iK
iK
$+\cdots$
A'
$\mathcal{F}$
$+\cdots$
$i\mathcal{M}$

Derivation

$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$ 

$+$ 

$+$ 

$+ \cdots$

Result

We conclude

$$C_L(E, \vec{P}) - C_\infty(E, \vec{P}) = -\sum_{n=0}^{\infty} A' F[-i\mathcal{M}F]^n A = -A' \frac{1}{F^{-1} + i\mathcal{M}} A$$

So for given values of $\{L, \vec{n}_P\}$, the energies in the spectrum are all E^* for which

$$\det(F^{-1} + i\mathcal{M}) = 0\,.$$