Progress Towards $\Delta I = 1/2 \; K \to \pi\pi$ Decays with G-parity Boundary Conditions

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Outline

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- Challenges
- G-Parity Implementation
- G-Parity Contractions
- The Strange Quark
- Conclusions and Outlook

Motivation

$\mathbf{K} \to \pi\pi$ Decays

- Direct CP-violation first observed in $K \to \pi\pi$ decays.
- Two types of decay:

$$\begin{aligned} \Delta I = 3/2 \ &: K^+ \to (\pi^+\pi^0)_{I=2} \quad \text{with amplitude } A_2 \\ \Delta I = 1/2 \ &: K^0 \to (\pi^+\pi^-)_{I=0} \\ & \quad\ K^0 \to (\pi^0\pi^0)_{I=0} \quad \text{with amplitude } A_0 \end{aligned}$$

- Direct CP-violation: $\epsilon' = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}} \left(\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right)$

  where

  $\omega = \mathrm{Re}A_2/\mathrm{Re}A_0$ and δ_I are strong rescattering phase shifts.

- Strong interactions very important – origin of the so-called $\Delta I = 1/2$ rule: preference to decay to $I = 0$ final state. Mechanism for this is not yet understood.

$\mathrm{K} \to \pi\pi$ Decays on the Lattice.

- Direct calculation of $K \to \pi\pi$ decays essential to understanding $\Delta I = 1/2$ rule and in the search for BSM physics.
- Lattice computation of realistic decays has only recently become possible.
- RBC & UKQCD recently published (arXiv:1111.1699) calculation of $\Delta I = 3/2$ decay using:
  - 2+1f domain wall fermions on a $32^3 \times 64 \times 32$ lattice with $a^{-1} = 1.37(1)\ \mathrm{GeV}.$
  - Near physical pions: $m_\pi^{PQ} \sim 140\ \mathrm{MeV}, m_\pi^{\mathrm{uni}} \sim 170\ \mathrm{MeV}$
  - Energy conserving decays
- Determined $\mathrm{Re}A_2$ and $\mathrm{Im}A_2$.

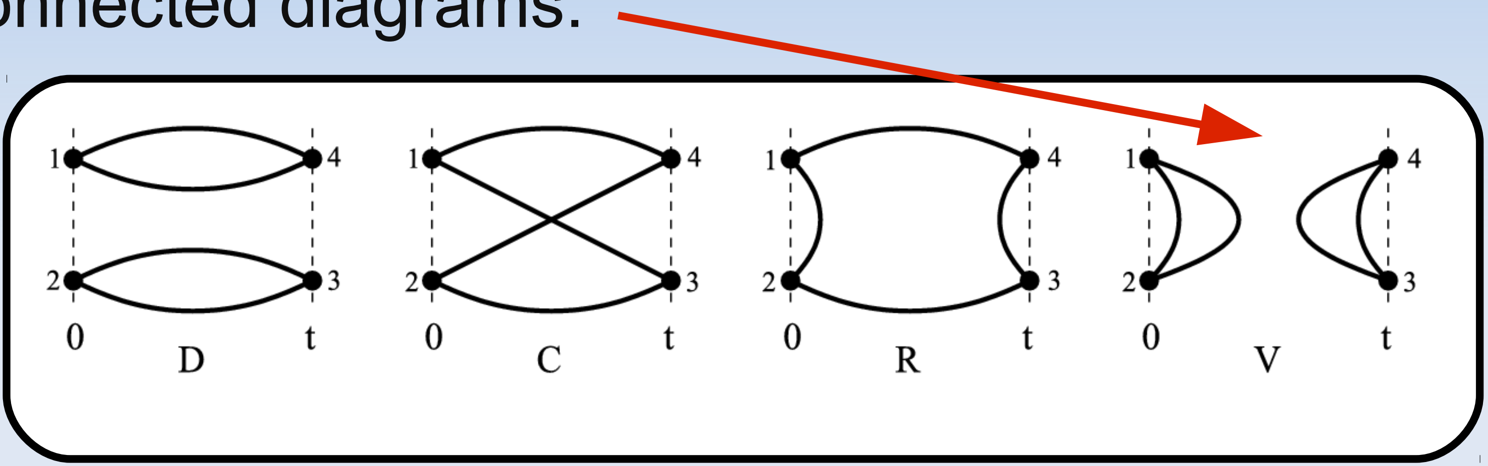
$K \to \pi\pi$ Decays on the Lattice.

- Combining results of $\Delta I = 3/2$ calculation with experimental value for ϵ'/ϵ, we obtained first value for $\mathrm{Im}A_0$
- Calculation of A_0 from first principles is much more difficult.

Challenges

Forming the $\pi\pi$ Propagator

- $\pi\pi$ state has vacuum quantum numbers, hence there are disconnected diagrams:



- Need large statistics and many source positions (or A2A/AMA propagators) to resolve.
- With Blue Gene/Q resources we can now perform such calculations with large enough physical volumes.

Physical Kinematics

- Best method is to use a stationary kaon and pions moving with equal momentum in opposite directions.
- This is an excited state of the $\pi\pi$ system – normally require very large statistics for decent signal.
- Instead impose antiperiodic BCs on d-quark propagator. $\pi^{\pm}$ gains momentum π/L, but for π^0 the p's cancel.
- This breaks isospin symmetry! However for $\Delta I = 3/2$ we can use Wigner-Eckart theorem and isospin to relate $\Delta I_z = 1/2\ K^+ \to \pi^+\pi^0$ to $\Delta I_z = 3/2\ K^+ \to \pi^+\pi^+$.
- As $\pi^+\pi^+$ is only charge-2 final state (q-conservation still true), isospin breaking becomes unimportant as this state cannot mix with other isospin states.
- Note: In practise we needed APBC in 2 dirs for physical kinematics.

Physical Kinematics

- For $\Delta I = 1/2$ the Wigner-Eckart trick cannot be used.
- If we stay with APBC on d-quarks, isospin-breaking would allow mixing between $I = 0$ and $I = 2$ final states. Separation would be difficult.
- I=0 state needs moving π^0 , but d-quark momentum cancels in $d\bar{d}$.
- Need to apply BCs that commute with isospin and produce moving π^0 as well as π^+ and π^-.
- Also, for $\Delta I = 1/2$ the vacuum plays a role, so the BCs must be applied to both the valence and sea quarks.

G-Parity Boundary Conditions

- G-parity is a charge conjugation followed by a 180 degree isospin rotation about the y-axis:

Wiese, Nucl.Phys.B375, (1992)

Kim, arXiv:hep-lat/0311003 (2003)

$$\hat{G} = \hat{C} e^{i\pi \hat{I}_y} \quad : \quad \begin{aligned} \hat{G}|\pi^{\pm}\rangle &= -|\pi^{\pm}\rangle \\ \hat{G}|\pi^{0}\rangle &= -|\pi^{0}\rangle \end{aligned}$$

- At the quark level:

$$\hat{G} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -C\bar{d}^T \\ C\bar{u}^T \end{pmatrix}$$ where $C = \gamma^2 \gamma^4$.

- G-parity commutes with isospin.
- Pions are all eigenstates with e-val -1, hence G-parity BCs make pion wavefunctions antiperiodic, with minimum momentum π/L.

G-Parity Implementation

Gauge Field Boundary Conditions

- d-field becomes $C\bar{u}^T$ across the boundary. Consider a bilinear on the boundary under a gauge transformation V :

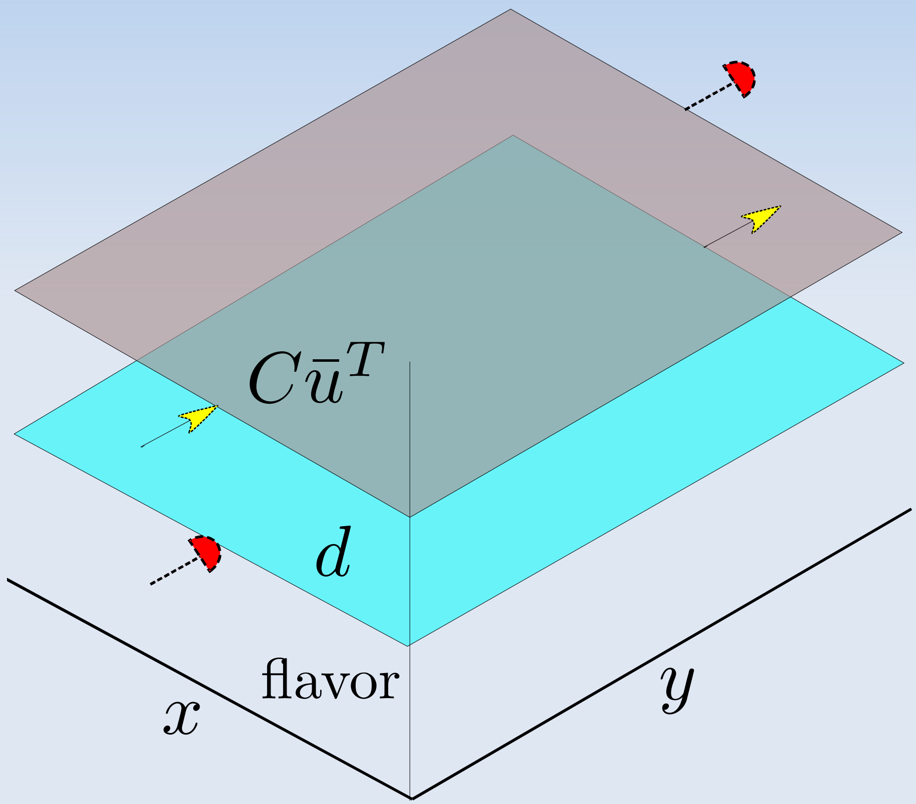
$$\bar{d}(L-1)U_y(L-1)C\bar{u}^T(0)$$
$$\longrightarrow \bar{d}(L-1)V^\dagger(L-1)U_y(L-1)V^*(0)C\bar{u}^T(0)\,.$$

- Link must transform as

$$U_y(L-1) \to V(L-1)U_y(L-1)V^T(0)$$

- Link parallel to boundary on on other side $(y \geq L)$ must then transform as:
- $U_x(x,y,..) \to V^*(x,y,..)U_x(x,y,..)V^T(x+1,y,..)$
- Gauge fields therefore obey complex-conjugate BCs.

The Two-Flavor Method



- Two fermion fields on each site indexed by flavor index:

$$\psi^{(1)}(x) = d(x),\ \psi^{(2)}(x) = C\bar{u}^T(x)$$

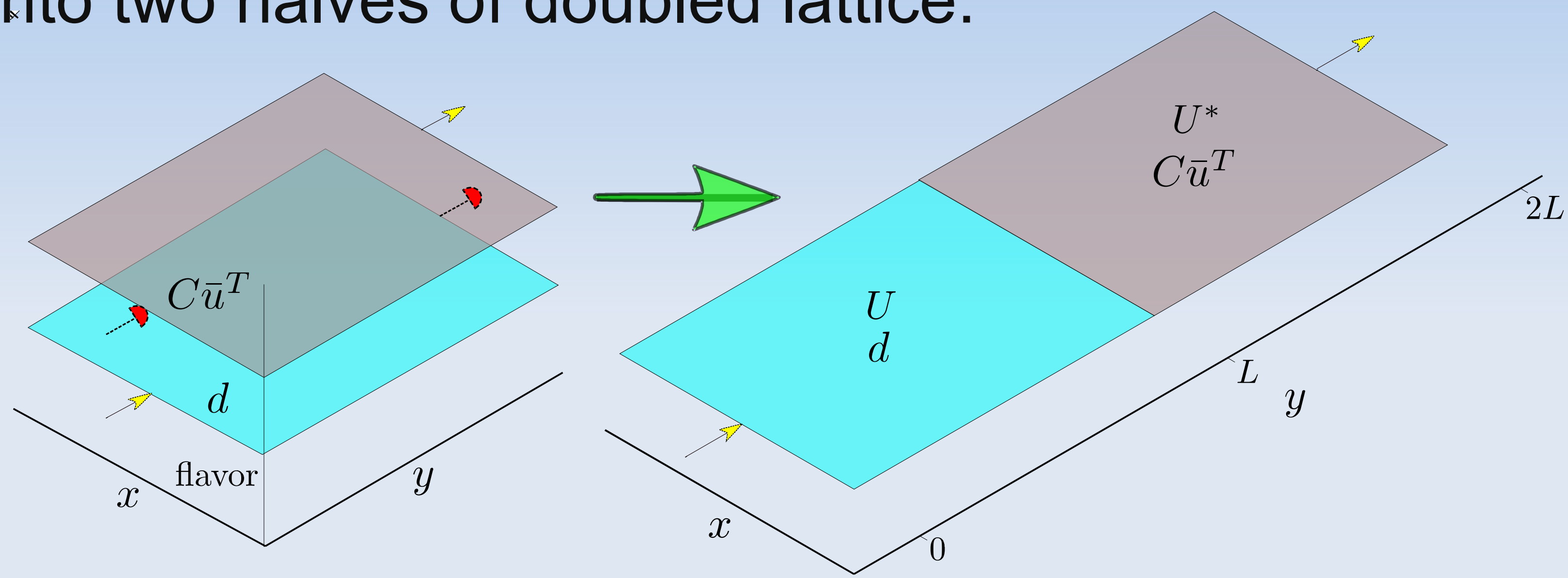
- BCs are:

$$\psi^{(1)}(x + L\hat{y}) = \psi^{(2)}(x),$$
$$\psi^{(2)}(x + L\hat{y}) = -\psi^{(1)}(x),$$

- Periodic BCs in other dirs.
- Single U-field shared by both flavors, with complex conj BCs.
- Dirac op for $\psi^{(2)}$ uses U_μ^*.

The One-Flavor Method

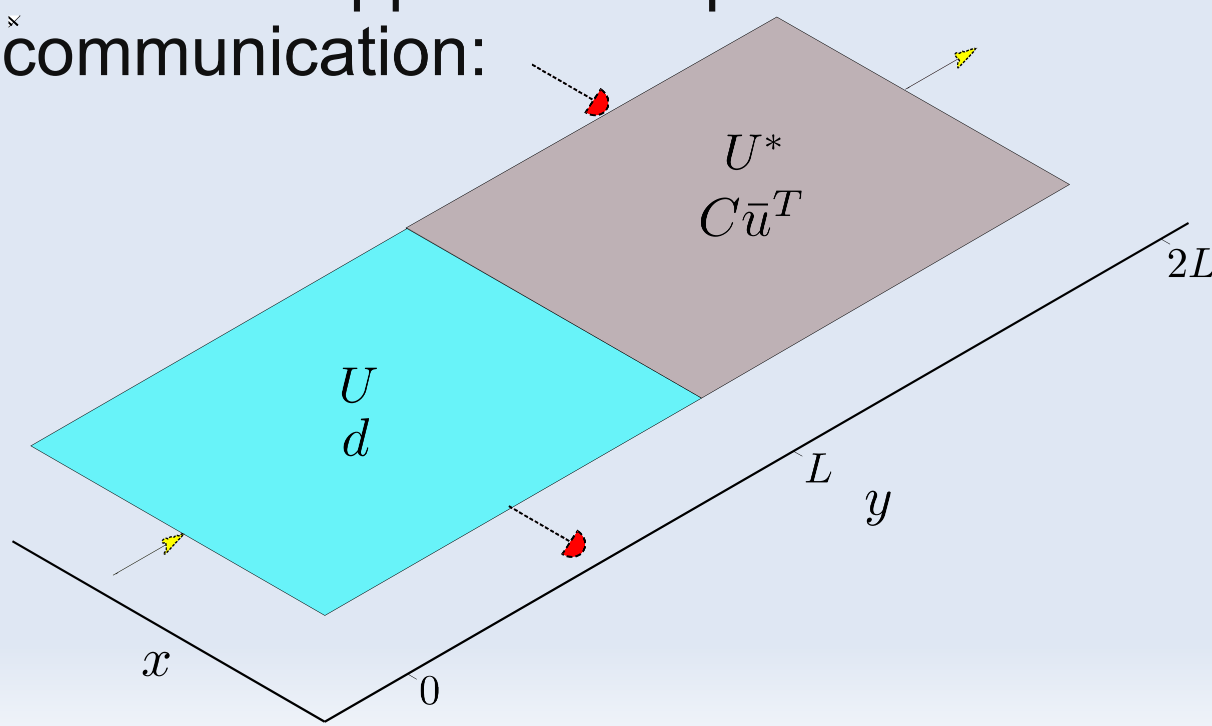
- Obtain equivalent formulation by unwrapping flavor indices onto two halves of doubled lattice:



- Antiperiodic boundary conditions in G-parity direction.
- U-field on first half and U^*-field on second half.

Choosing an Approach

- One flavor setup is much easier to implement.
- However recall that we needed APBC in 2 directions for physical kinematics in $\Delta I = 3/2$ calculation.
- G-parity in >1 dir using one-flavor method requires doubling the lattice again, which is highly inefficient.
- A second approach requires non-nearest neighbour communication:



- Also inefficient depending on machine architecture.
- Choose to implement two-flavor method.

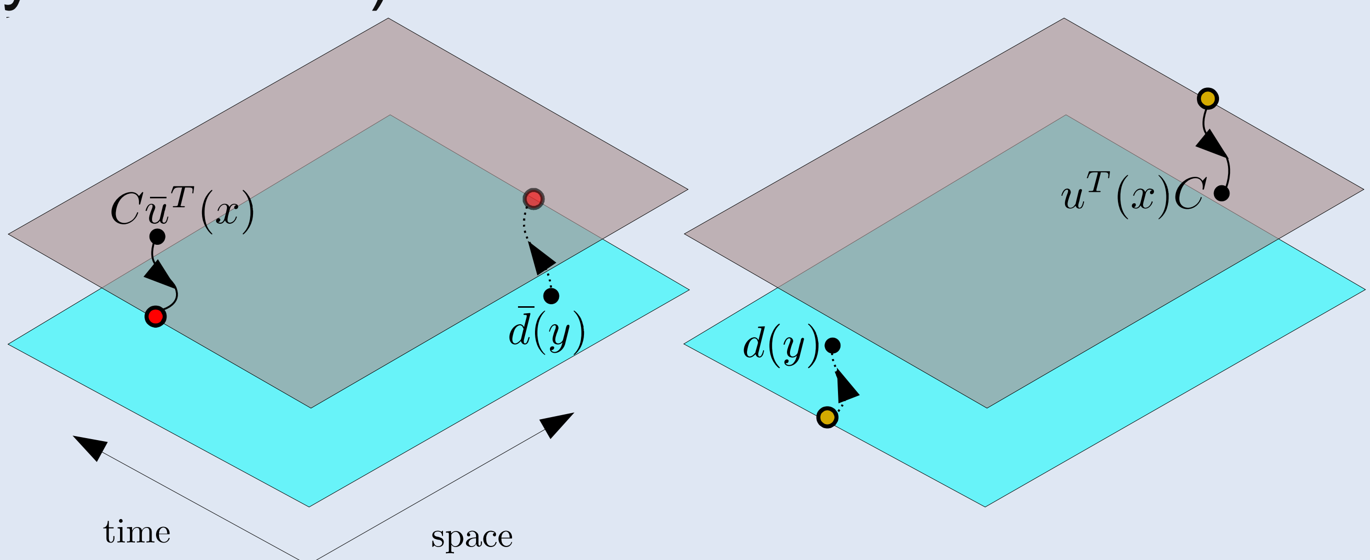
G-Parity Contractions

Unusual Contractions

- Flavor mixing at boundary allows contraction of up and down fields:

$$\psi^{(2)}_x \bar{\psi}^{(1)}_y = \mathcal{G}^{(2,1)}_{x,y} = C\bar{u}^T_x \bar{d}_y \, ,$$
$$\psi^{(1)}_y \bar{\psi}^{(2)}_x = \mathcal{G}^{(1,2)}_{y,x} = -d_y u^T_x C^T$$

- Interpret as boundary creating/destroying flavor (violating baryon number):



- Also have γ^5-hermiticity: $\left[\gamma^5 \mathcal{G}^{(2,1)}_{x,y} \gamma^5\right]^\dagger = \mathcal{G}^{(1,2)}_{y,x}$

Exploiting the Underlying Gauge-Field Symmetry

- Quarks on flavor-1 plane interact with U field, and those on flavor-2 plane with U*.
- Suggests propagators are related in some way.
- In fact, we find that:

$$\begin{aligned} \mathcal{G}_{x,z}^{(2,2)} &= -\gamma^5 C \left[\mathcal{G}_{x,z}^{(1,1)}\right]^* C\gamma^5 \\ \mathcal{G}_{x,z}^{(1,2)} &= +\gamma^5 C \left[\mathcal{G}_{x,z}^{(2,1)}\right]^* C\gamma^5 \end{aligned}$$

- Relative sign due to – sign at boundary between u and d.
- Substantially simplifies contractions.
- In some cases these relations can be used to reduce the number of propagator inversions required.

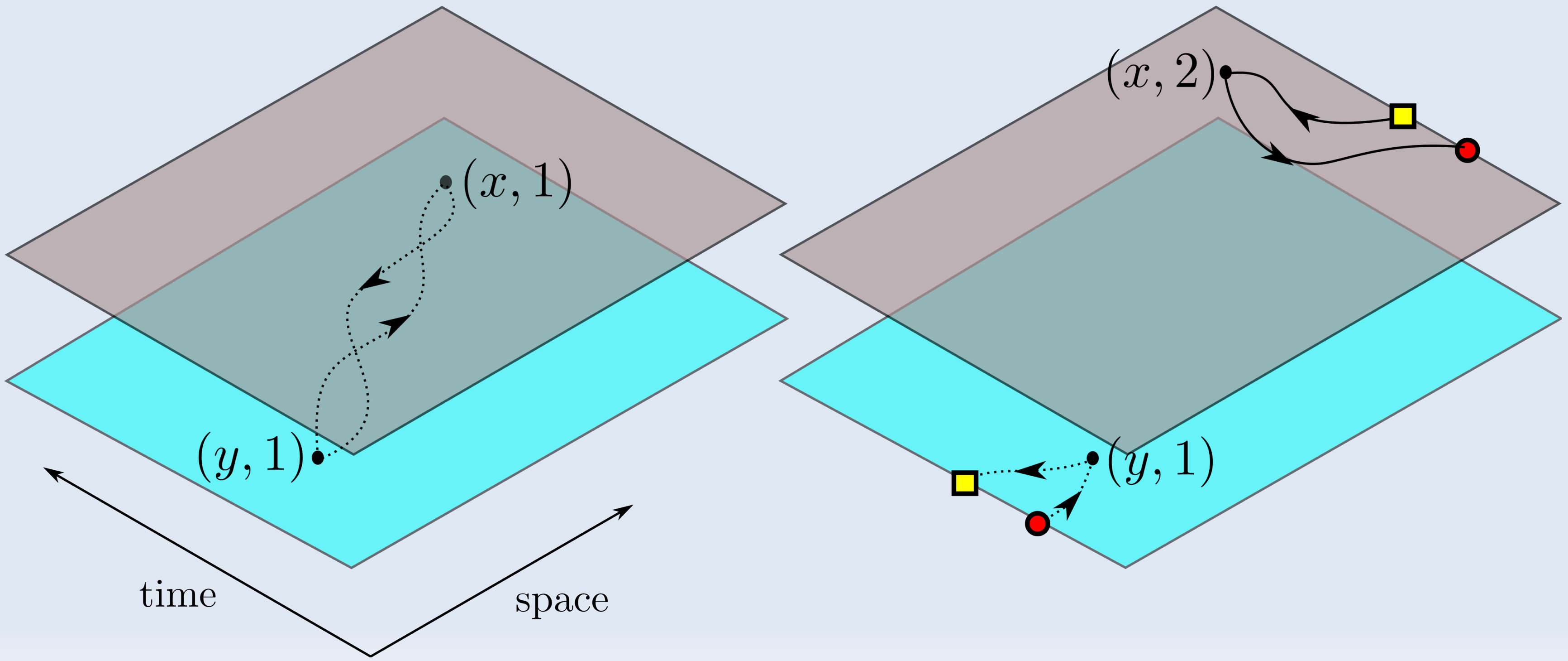
Pion Correlation Functions

- π^+ correlation function

$$\langle \bar{d}_x \gamma^5 u_x \bar{u}_y \gamma^5 d_y \rangle = \langle \bar{\psi}^{(1)}_x [\gamma^5 C] \bar{\psi}^{(2)\,T}_x \psi^{(2)\,T}_y [C\gamma^5] \psi^{(1)}_y \rangle$$

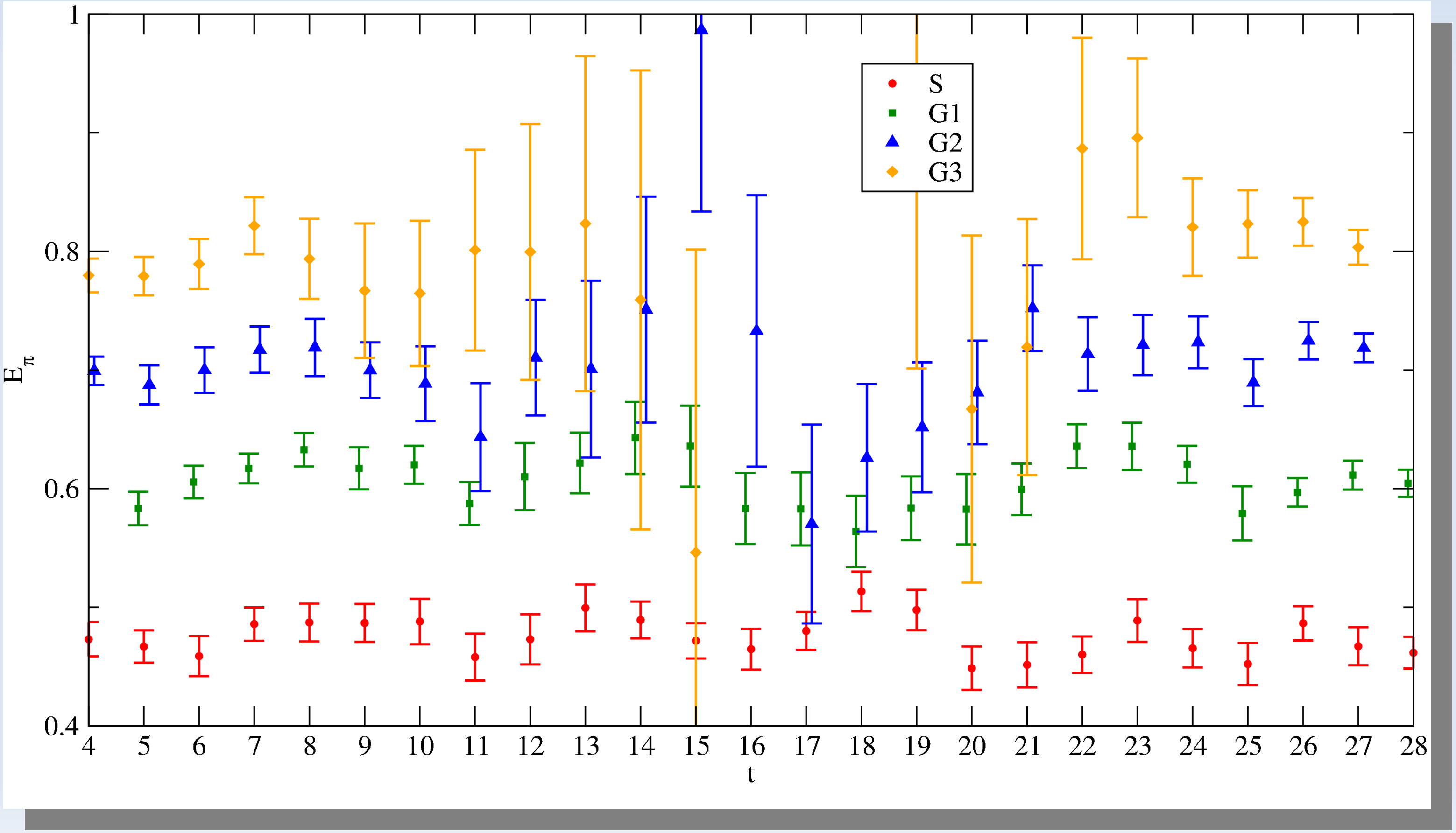
- Now has *two* contractions:

$$\mathrm{tr}\left\{ \mathcal{G}^{(1,1)\,\dagger}_{x,y} \mathcal{G}^{(1,1)}_{x,y} \right\} - \mathrm{tr}\left\{ \mathcal{G}^{(2,1)\,\dagger}_{x,y} \mathcal{G}^{(2,1)}_{x,y} \right\}$$

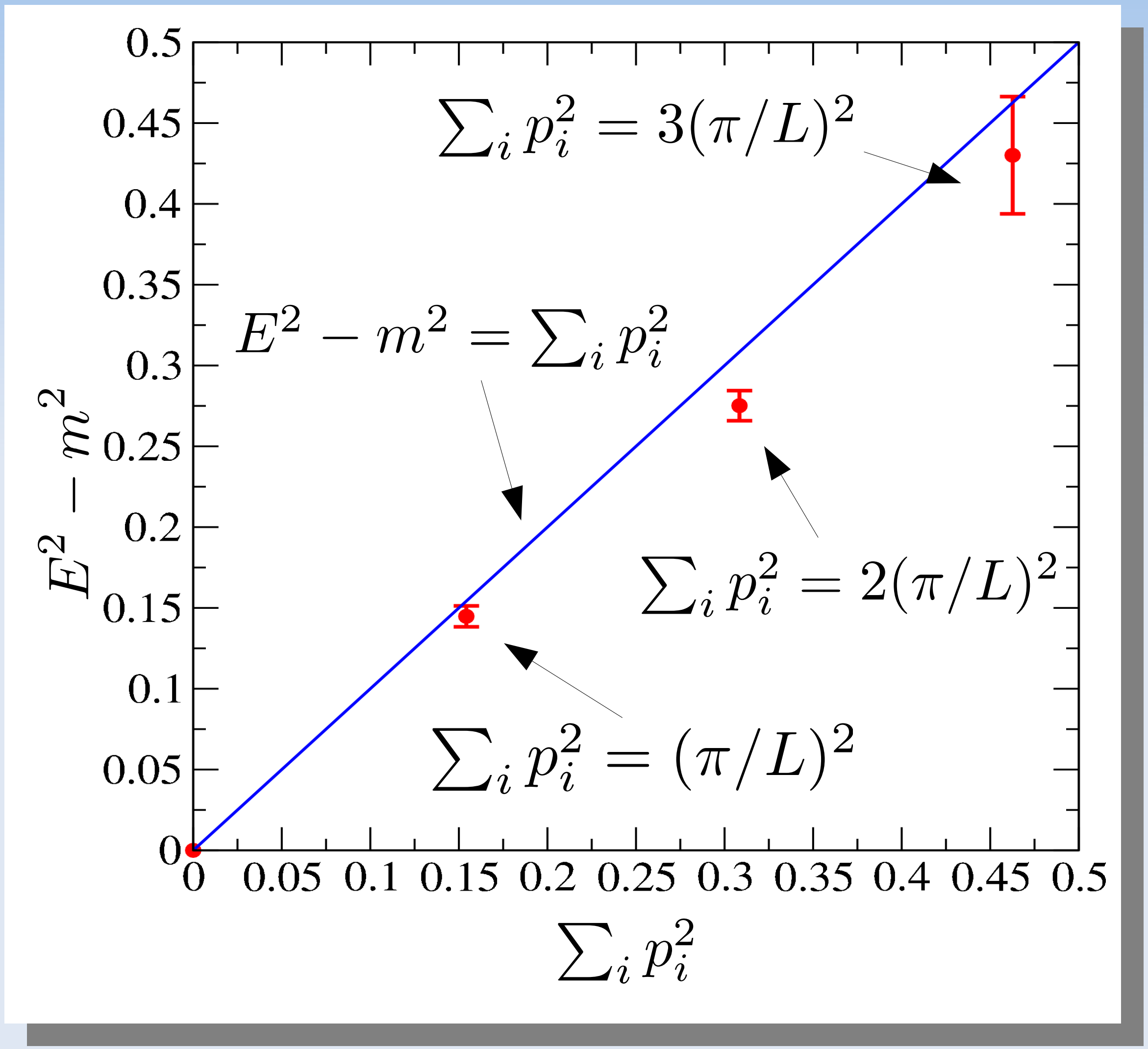


Results: Pion Correlator

- Generated an $8^3 \times 32 \times 10$ DWF quenched ensemble.
- ~150 configs (20 MD tu's sep) with G-parity in 0,1,2,3 dirs.
- Coulomb-gauge fixed wall source propagators.



Results: Pion Dispersion Relation



- Deviations from continuum disp. reln. expected on lattice. e.g. free-field: $E^2 - m^2 = \sum_i \sin^2(p_i)$

The Strange Quark

Kaons

- $K \to \pi\pi$ calculation needs stationary K^0.
- $\frac{1}{\sqrt{2}}(\bar{s}d + \bar{d}s)$ not a G-parity eigenstate.
- Need an eigenstate with e-val +1 for periodic BCs and hence $p_{\mathrm{min}} = 0$.
- Introduce 'strange isospin' (I'): s-quark in doublet $\begin{pmatrix} s' \\ s \end{pmatrix}$
- A neutral kaon-like state:

$$K_0' = \frac{1}{2}(\bar{s}d + \bar{d}s + \bar{s}'u + \bar{u}s')$$

is an eigenstate of 'modified G-parity': $\hat{G} = \hat{C} e^{i\pi \hat{I}_y} e^{i\pi \hat{I}_y'}$ with e-val +1.

- Need factor of 2 in decay calc as only 1/2 of components of initial kaon couple to $\pi\pi$.

Locality

- Theory has one too many flavors. Must take square-root of s'/s determinant in evolution to revert to 3 flavors.
- Determinant becomes non-local.
- Non-locality is however only a boundary effect that vanishes as $L \to \infty$. With sufficiently large volumes the effect should be minimal.
- Estimate size of effect?

  - Staggered ChPT?

  - Observe effect of changing from $d \to C\bar{u}^T \to -d$ to $d \to C\bar{u}^T \to +d$ for which $\sqrt{\mathrm{Det}(D)}$ is local (= $\mathrm{Pfaffian}(D)$)?

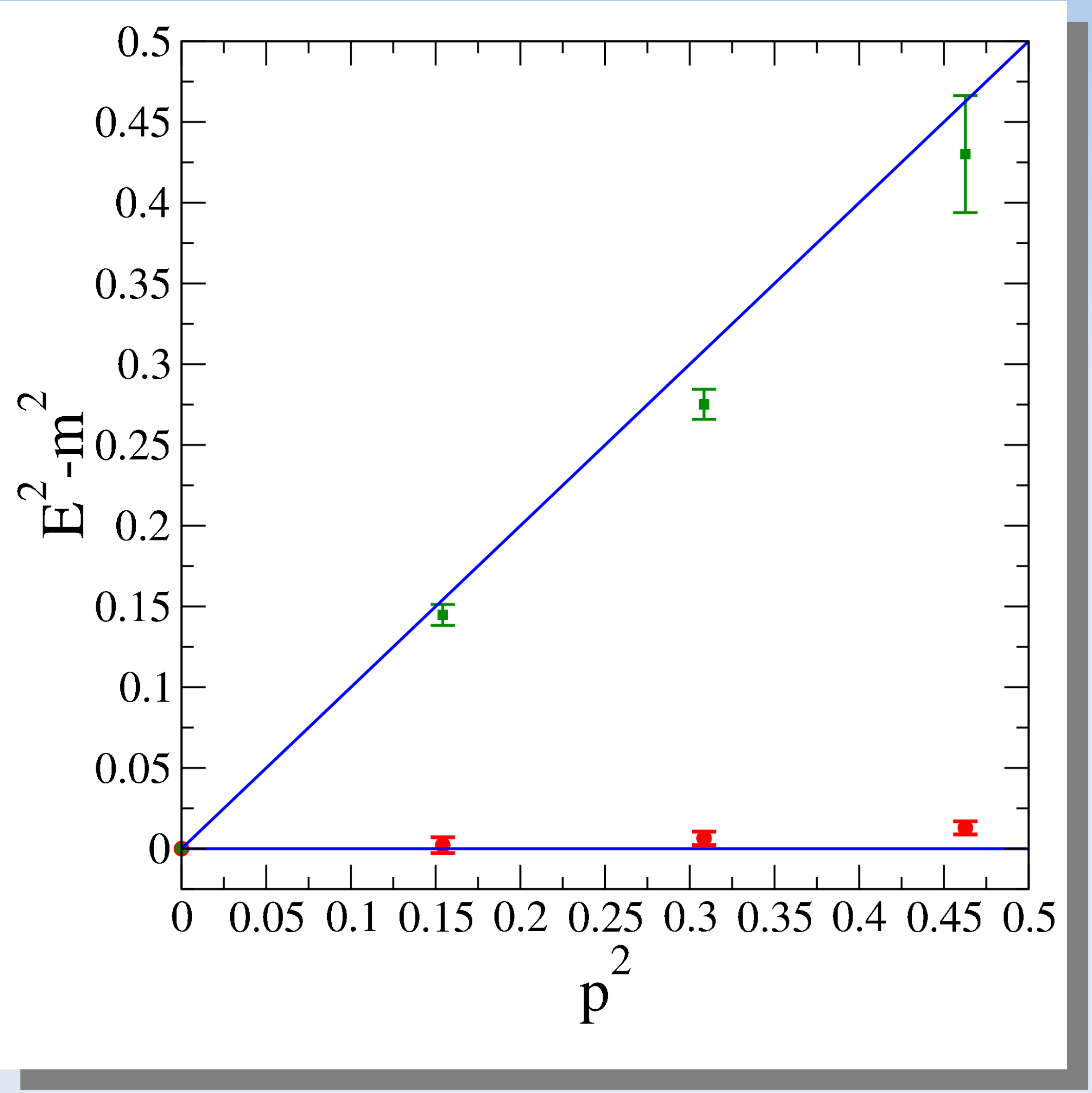
Charged Kaon Correlator

- K^+ analogue: $|K^{+\prime}\rangle = \frac{1}{\sqrt{2}}(\bar{u}\gamma^5 s - \bar{s}'\gamma^5 d)|0\rangle$
- 2-point function also has 4 contractions:
  (flavour indices $3 = s$, $4 = C\bar{s}'^T$):

$$\frac{1}{2}\mathrm{tr}\left\{\mathcal{G}_{x,y}^{(3,3)\,\dagger}\mathcal{G}_{x,y}^{(1,1)}\right\} + \frac{1}{2}\mathrm{tr}\left\{\mathcal{G}_{x,y}^{(3,3)}\mathcal{G}_{x,y}^{(1,1)\,\dagger}\right\}$$
$$+\frac{1}{2}\mathrm{tr}\left\{\mathcal{G}_{x,y}^{(4,3)\,\dagger}\mathcal{G}_{x,y}^{(2,1)}\right\} + \frac{1}{2}\mathrm{tr}\left\{\mathcal{G}_{x,y}^{(4,3)}\mathcal{G}_{x,y}^{(2,1)\,\dagger}\right\}$$

- If we make the masses of the (s', s) and (u, d) doublets the same this is just the π^+ correlation function but with the *opposite sign* between the contractions.
- Periodicity of spatial dependence appears to arise due to some cancellation between the two contractions.

Results: Degenerate $K^{+\prime}$ Dispersion Relation



Conclusions and Outlook

Conclusions and Outlook

- G-parity boundary conditions look to be very promising for realising $\Delta I = 1/2$ $K \to \pi\pi$ decays with physical kinematics.
- Direct calculation of A_0 is essential for understanding $\Delta I = 1/2$ rule and in the search for BSM physics.
- Coding two-flavor method is challenging but significant progress has been made.
- Aim to dedicate significant BlueGene/Q resources towards generating $32^3 \times 64 \times 32$ DWF Iwasaki+DSDR ensembles with G-parity BCs and physical pions.