Thermodynamics of the unitary Fermi gas

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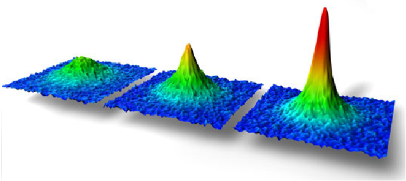
What is a Fermi gas?

A Fermi gas is a dilute system of fermions

Examples:

- electrons inside a metal
- quark-gluon plasma in the inner core of a neutron star
- atomic gas e.g. of ^{40}K or ^{6}Li atoms ($\Rightarrow$ experiment!)

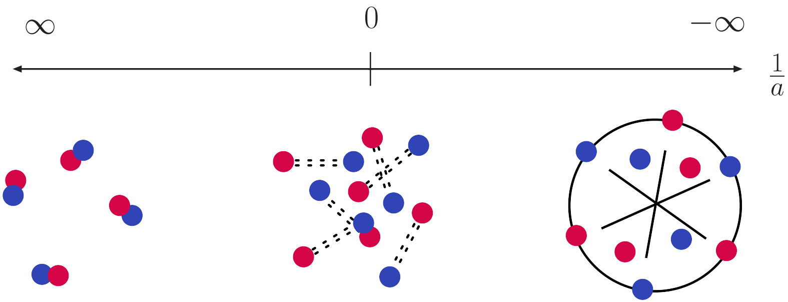


experiment by the JILA group (University of Colorado)

low temperature: quantum phenomena become important
identical fermions $\Rightarrow$ no s-wave scattering $\Rightarrow$ ideal gas model

What is a unitary Fermi gas?

Fermions of two species ⇒ interactions become important
Low-energy interactions are characterised by the scattering length *a*



BEC regime	UNITARITY	BCS regime
strongly bound (bosonic) molecules of two fermions	strongly interacting fermions	pairs of fermions weakly bound in momentum space

What is interesting about unitarity?

- System is dilute (range of potential $\ll$ interparticle distance) and strongly interacting (interparticle distance $\ll$ scattering length) at the same time
- No length scales associated with interactions $\Rightarrow$ universal behaviour
- Only relevant parameters: temperature and density
- High-temperature superfluidity

neutron star	$T_c = 10^6$K	$T_c = 10^{-5}\,T_F$
high-T_c superconductor	$T_c = 10^2$K	$T_c = 10^{-3}\,T_F$
atomic Fermi gas	$T_c = 10^{-7}$K	$T_c = 10^{-1}\,T_F$

- Experimental data available

What is interesting about unitarity?

Strong interactions $\Rightarrow$ No small parameter for perturbation theory

No exact theory for Fermi gas at unitarity!

What to do?

- Approximate schemes (e.g. mean-field theory) involve uncontrolled approximations
- Numerical Methods
  $\Longrightarrow$ Good results for critical temperature and other quantities

Project: Equation of state and temperature dependence of the contact density of the unitary Fermi gas near the critical point
Method: Determinant Diagrammatic Monte Carlo (DDMC) algorithm [Burovski, Prokof'ev, Svistunov, Troyer (2006)]

The Fermi-Hubbard model

Simplest lattice model for two-particle scattering

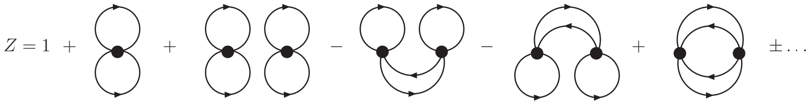
- Non-relativistic fermions
- Contact interaction between spin up and spin down
- On-site attraction $U < 0$ tuned to describe unitarity
- Grand canonical ensemble
- Finite 3D simple cubic lattice, periodic boundary conditions
- Continuum limit can be taken by extrapolation to zero density

$$H = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U \sum_{\mathbf{x}} c^\dagger_{\mathbf{x}\uparrow} c_{\mathbf{x}\uparrow} c^\dagger_{\mathbf{x}\downarrow} c_{\mathbf{x}\downarrow},$$

where $\epsilon_{\mathbf{k}} = \frac{1}{m} \sum_{j=1}^{3} (1 - \cos k_j)$ is the discrete FT of $\frac{-\nabla^2}{2m}$.

Finite temperature formalism

Grand canonical partition function in imaginary time interaction picture: $Z = \mathrm{Tr} e^{-\beta H}$:



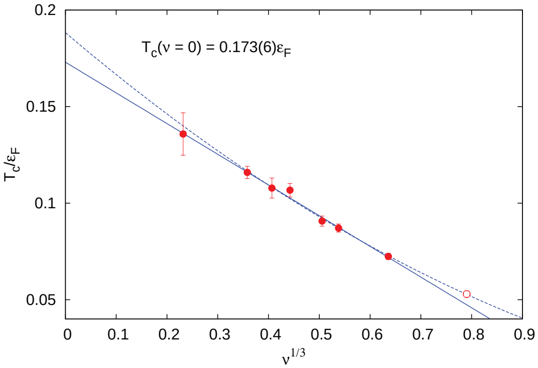
Sign problem!

The diagrams of each order can be written as the product of two matrix determinants [Rubtsov, Savkin, Lichtenstein (2005)]

$$Z = \sum_{p, S_p} (-U)^p \det \mathbf{A}^{\uparrow}(S_p) \det \mathbf{A}^{\downarrow}(S_p),$$

where S_p is the vertex configuration and the matrix entries are free (finite temperature) propagators

Continuum limit



physical scale $\nu = nb^3$

- $\nu = \langle \sum_\sigma c^\dagger_{\mathbf{x}\sigma} c_{\mathbf{x}\sigma} \rangle$ dimensionless filling factor
- n particle density
- b lattice spacing

- continuum limit $b \to 0$ corresponds to $\nu \to 0$
- leading order lattice corrections $\propto b \propto \nu^{1/3}$

Results at T_c

Results at the critical point:

$$\begin{aligned} T_c/\varepsilon_F &= 0.171(5) \\ \mu/\varepsilon_F &= 0.429(7) \\ E/E_{\mathrm{FG}} &= 0.440(15) \\ \mathcal{C}/\varepsilon_F^2 &= 0.1101(9) \end{aligned}$$

and generalisation to the imbalanced case ($\mu_\uparrow \neq \mu_\downarrow$) for small imbalances

[Phys. Rev. A **82**, 053621 (2010), Lattice Proceedings 2009, 2010]

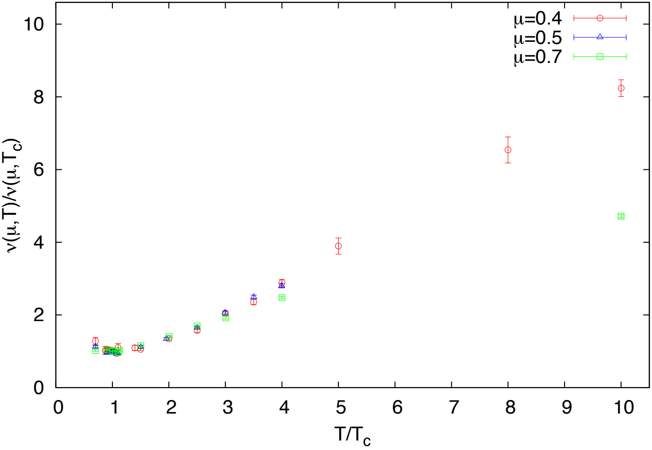
Temperatures beyond T_c

- The lattice critical temperature $T_c(\mu)$ for different lattice μ is known from previous work
- Use the lattice T_c as a benchmark to fix the temperature scale
- Simplest approach: hold lattice μ fixed and vary the lattice T
- Set lattice spacing independent of temperature:

$$b(\mu, T) = b(\mu, T_c) = \left(\frac{\nu(\mu, T_c)}{n(\mu, T_c)}\right)^{1/3}$$

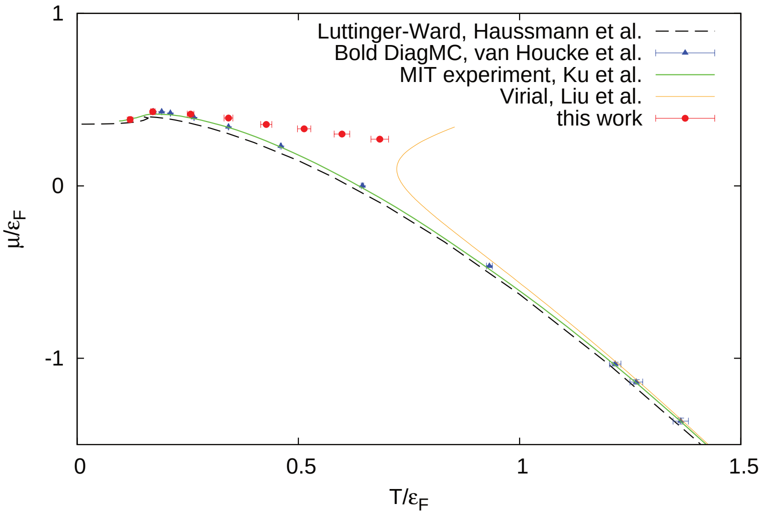
- Fix ratio $r = T(\mu)/T_c(\mu)$ for each lattice μ and perform continuum extrapolations as at T_c

Temperatures beyond T_c

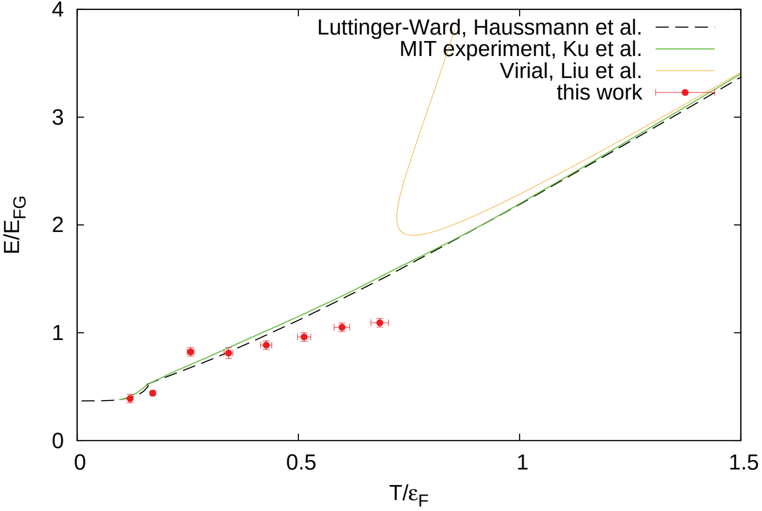


This approach works sufficiently close to the critical point.

Preliminary results: chemical potential



Preliminary results: energy per particle



Contact density

The contact can be interpreted as a measure for the local pair density [Tan (2008), Braaten (2010)].

Definition contact [Werner and Castin (2010)]:

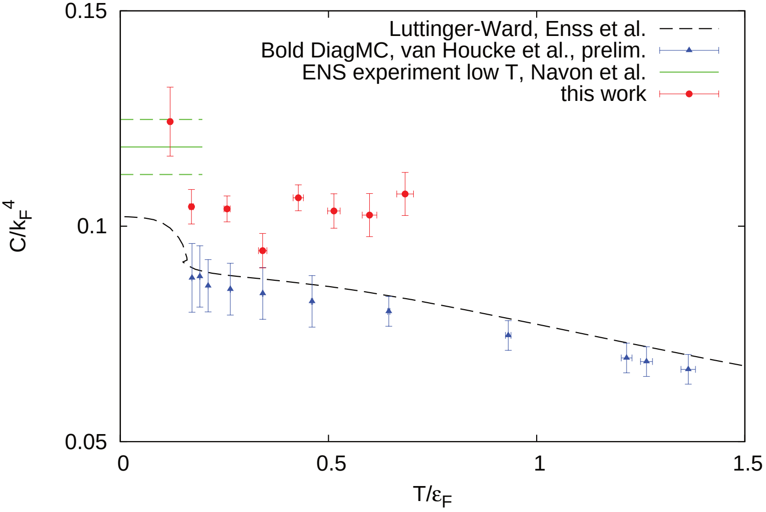
$$C = m^2 g_0 E_{\text{int}},$$

where g_0 is the physical coupling constant.

The contact density is $\mathcal{C} = C/V$ and has units k_F^4.

Expressed through lattice quantities: $\mathcal{C}/k_F^4 = \frac{UE_{\text{int}}}{4L^3 k_F^4}$

Preliminary results: contact density



Conclusions

- Lattice Field Theory is a useful tool for studying strongly interacting systems in condensed matter physics
- The DDMC algorithm can be applied to calculate the critical temperature of the Fermi gas at unitarity
- Generalisation to temperatures above and below T_c
- Temperature dependence of the chemical potential, the contact density and the energy density