Universal four-component Fermi gas on the lattice

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Lattice 2012

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Outline

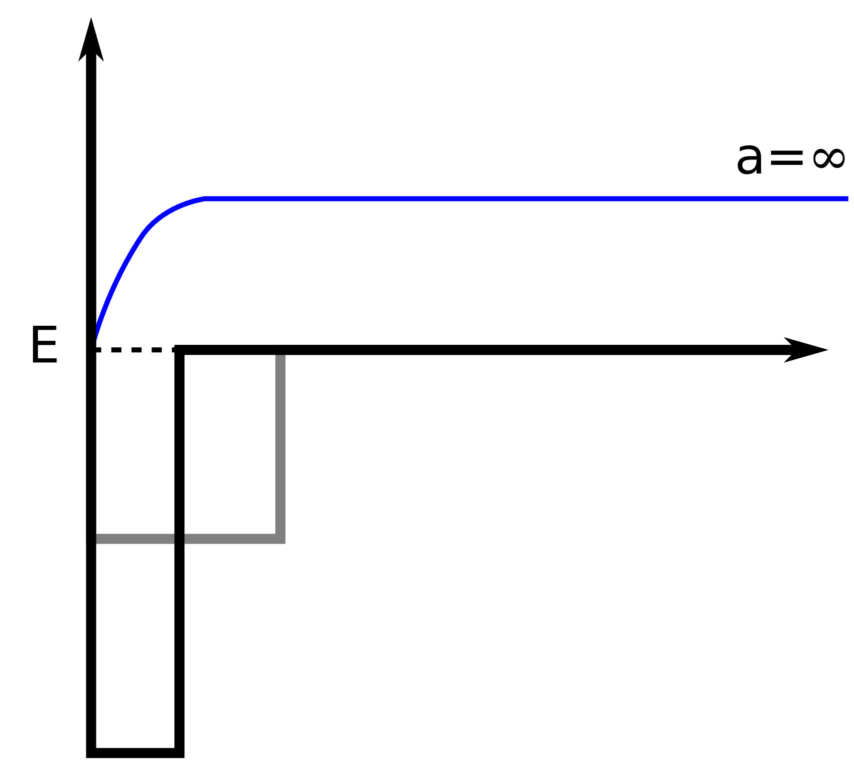
- Background (briefly)
- Description of the lattice theory
- Mean-field analysis of the model

Lattice theory for nonrelativistic fermions in one spatial dimension
Michael G. Endres, (2012) [arXiv:1204:6182].
(*accepted for publication in Phys. Rev. A*)

Unitary fermions

- Fermi gases have been the subject of intense interest during the past ten years, particularly unitary fermions
  - two-component gas
  - attractive interparticle potential
  - infinite scattering length, vanishing effective range

Properties of the many-body system only depend on the scattering length and density



Unitary fermions

- Physical realizations:
  - relevant for neutron stars?
  - ultra-cold atom experiments (exploiting Feshbach resonances)

Fermi gases in lower dimensions

- Fermi gases in lower and mixed dimensions also exhibit universal properties:
  - "Universal Fermi gases in mixed dimensions"
    Y. Nishida and D.T. Son
    Phys. Rev. Lett. **101**, 170401 (2008)
  - "Universal four-component Fermi gas in one dimension"
    Y. Nishida and D.T. Son
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Effective field theory considered by Son & Nishida

$$S_{cont}(\mu) = \int d^2x \left[\psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi - g(\psi^\dagger\psi)^4\right]$$

1+1 dimensions

$$\psi = (\psi_a, \psi_b, \psi_c, \psi_d)$$

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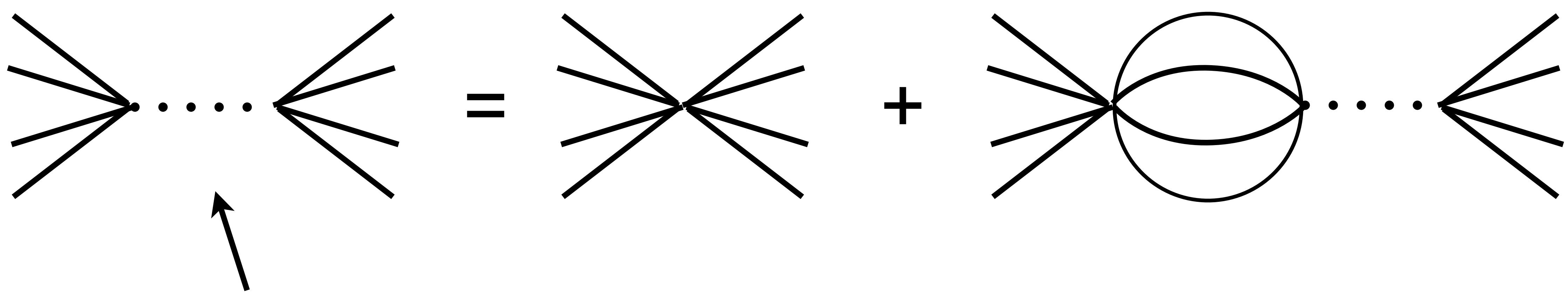
Relationship between coupling (g) and "scattering length":



four-particle
scattering amplitude

$$\mathcal{A}^{-1}(p)\big|_{p\to 0} = \frac{m}{4\pi a}$$

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Relationship between coupling (g) and "scattering length":

$\boxed{\text{Infinite scattering length: zero-energy four-body bound state}}$

$$\mathcal{A}^{-1}(p)\Big|_{p\to 0} = \frac{m}{4\pi a}$$

Properties of the model

This one-dimensional model exhibits many features in common with the unitary Fermi gas in three dimensions:

- One-dimensional analog of BCS-BEC "cross-over"
- Universal few- and many-body physics:
    - Effimov physics
    - analog of "Bertch parameter" and gaps for fermion excitations
- Universal relations, contact density (Shina Tan)
- Operator-state correspondence

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Nonperturbative physics

Path integral formulation

$$-S(\mu) = \sum_{\mathbf{n}\in\Lambda}\left[-\psi^\dagger_{\mathbf{n}}\left(1+\frac{1}{m}\right)\psi_{\mathbf{n}} + \psi^\dagger_{\mathbf{n}} e^{\mu}\psi_{\mathbf{n}-\mathbf{e}_0} + \psi^\dagger_{\mathbf{n}}\frac{1}{2m}\left(\psi_{\mathbf{n}+\mathbf{e}_1}+\psi_{\mathbf{n}-\mathbf{e}_1}\right) + g\left(\psi^\dagger_{\mathbf{n}} e^{\mu}\psi_{\mathbf{n}-\mathbf{e}_0}\right)^4\right]$$

$$Z(\mu) = \int [d\psi^\dagger][d\psi] e^{-S(\mu)}$$

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$$g^{1/4} \phi \psi^\dagger e^{\mu} \psi$$

(e.g., $\varphi = z_4$ field)

Conventional Hubbard-Stratonovich transformation results in a sign problem (after integrating out fermions)

Path integral formulation

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Solution: consider a different representation for the partition function based on a hopping parameter expansion

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Ignore interaction to begin with...

Consider a hopping parameter expansion

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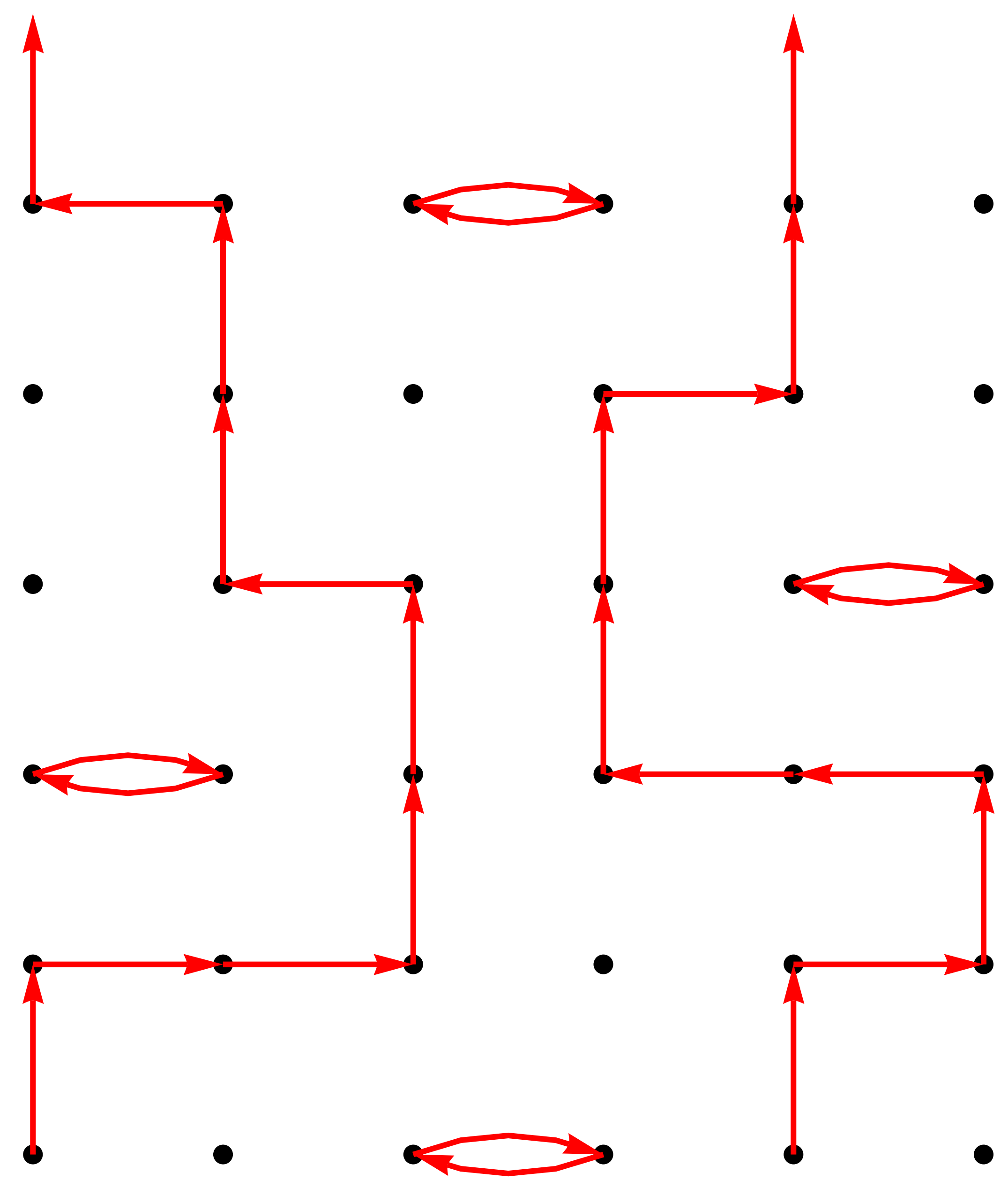
$$Z(\mu) = \int [d\psi^\dagger][d\psi] e^{-S(\mu)}$$

Expand in powers of the action, integrate out the fermions term by term



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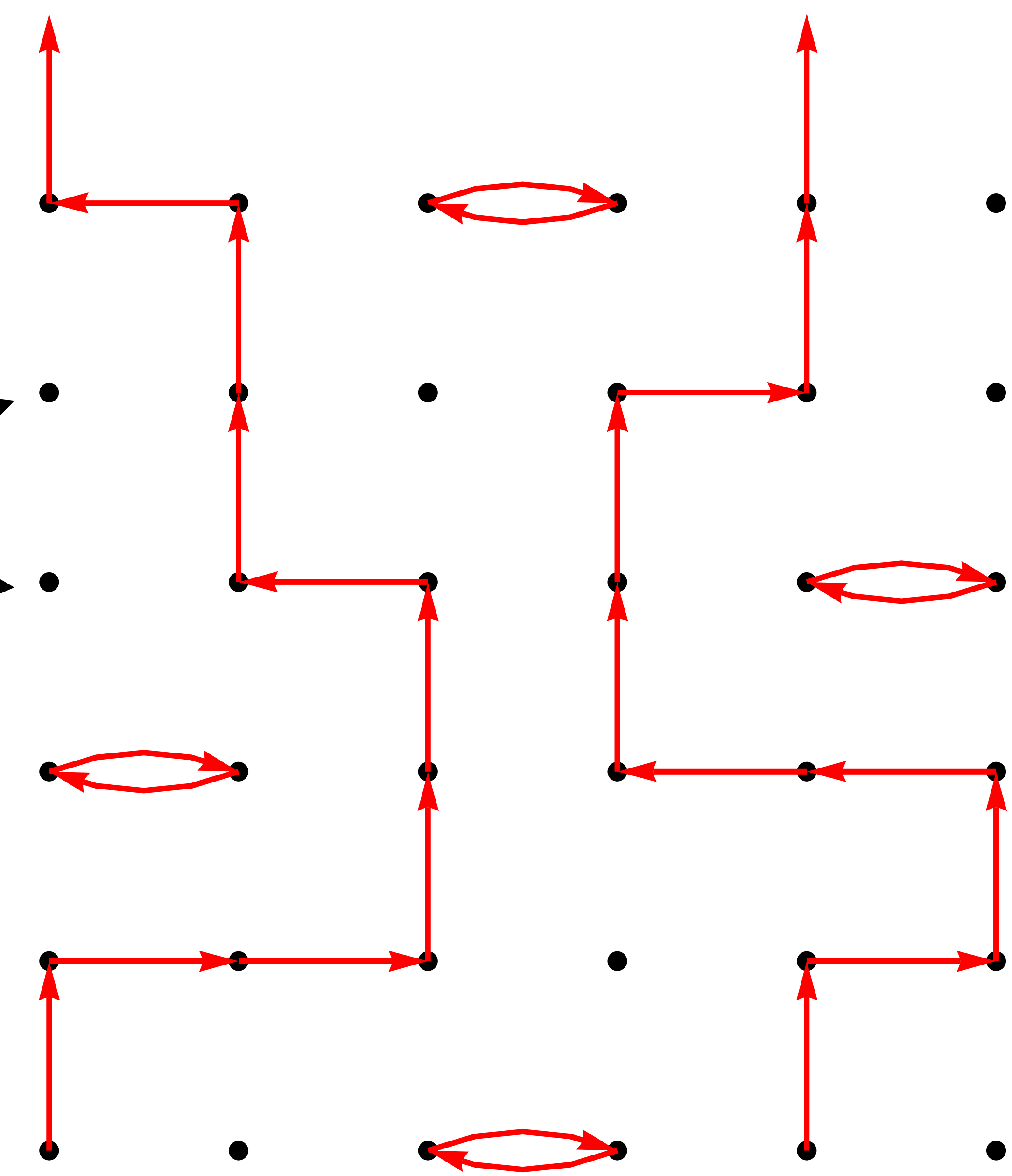
$$Z(\mu) = \int [d\psi^\dagger][d\psi] e^{-S(\mu)}$$

$$-\left(1 + \frac{1}{m}\right)$$



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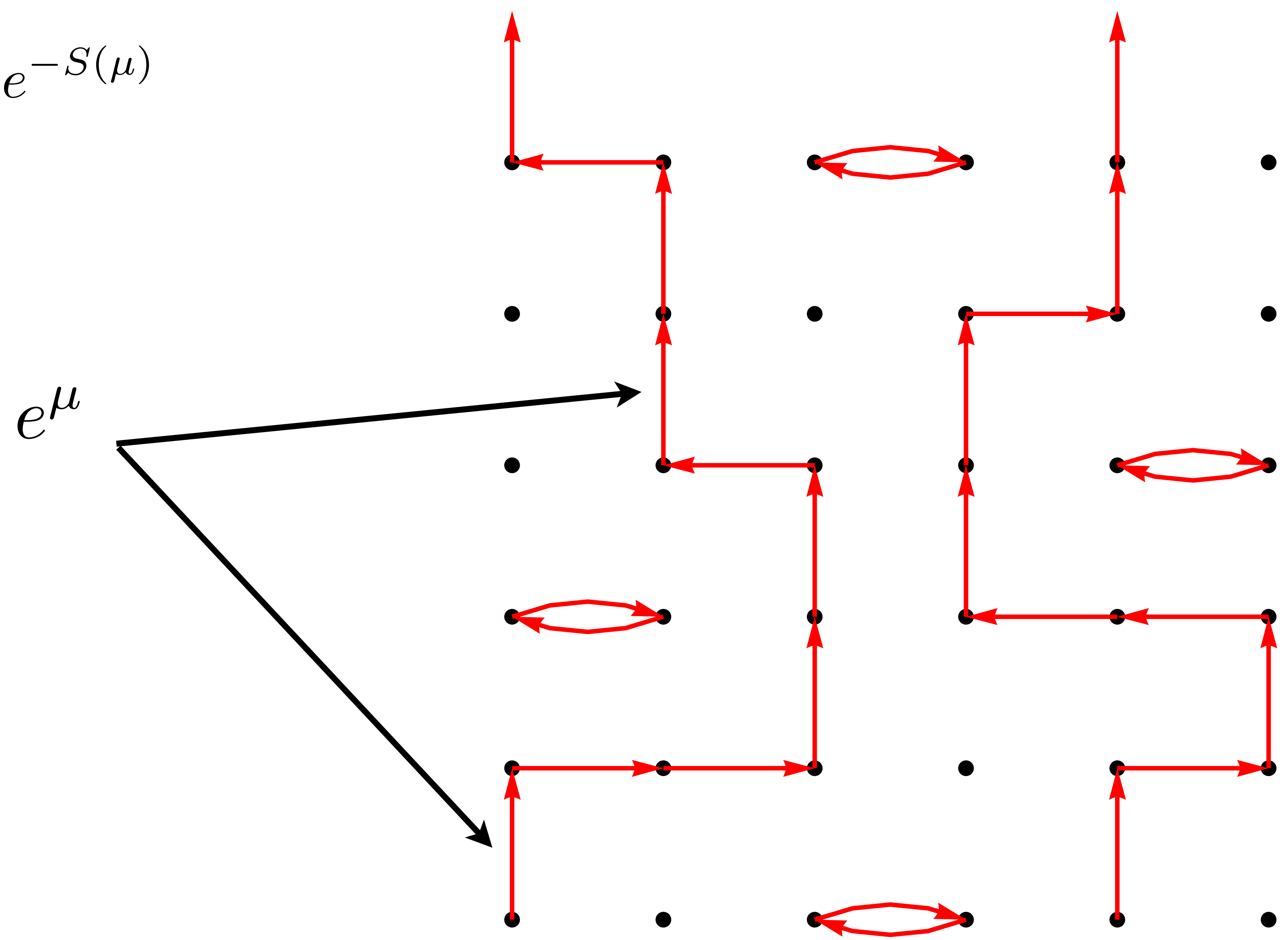

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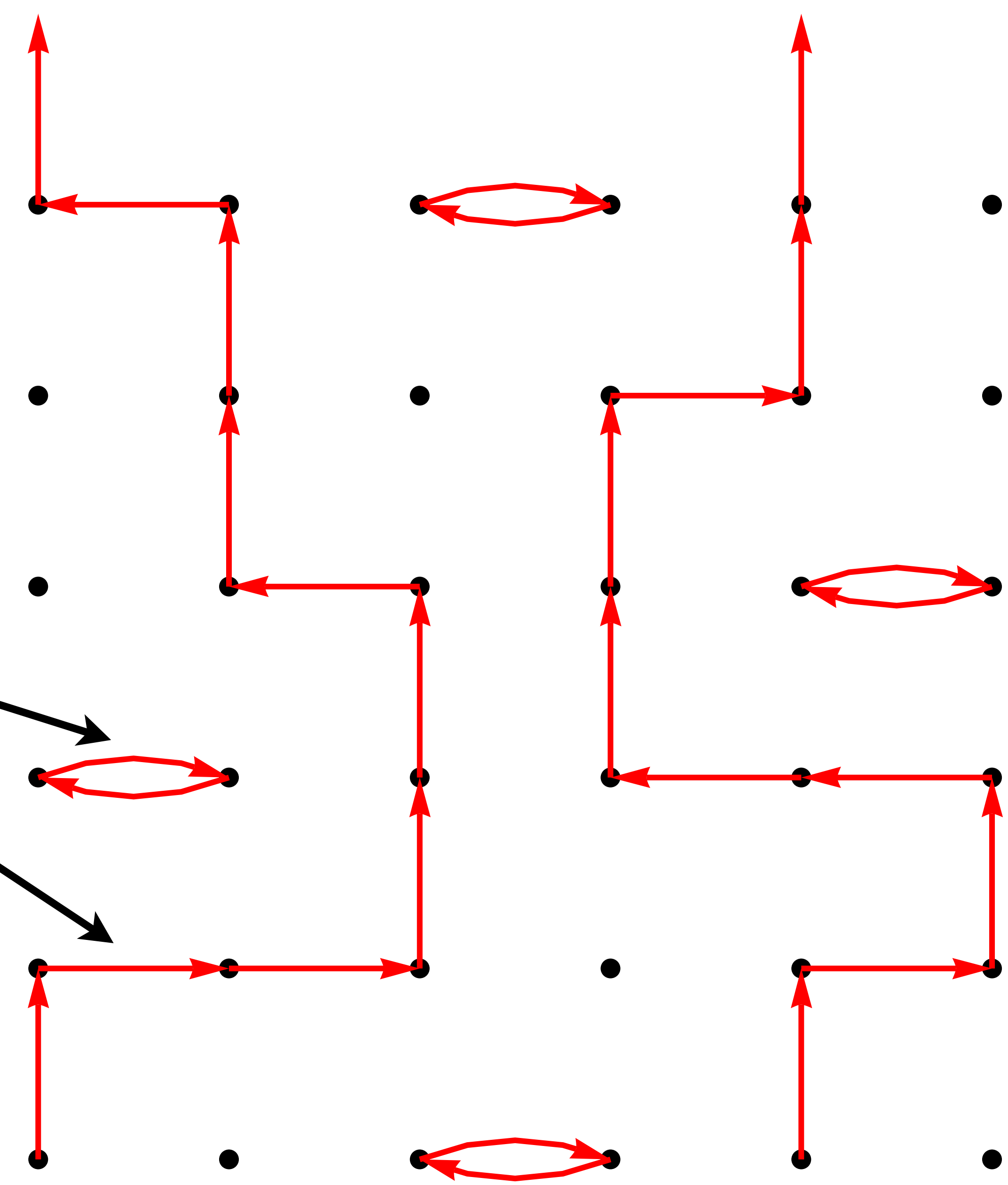
$$\frac{1}{2m}$$



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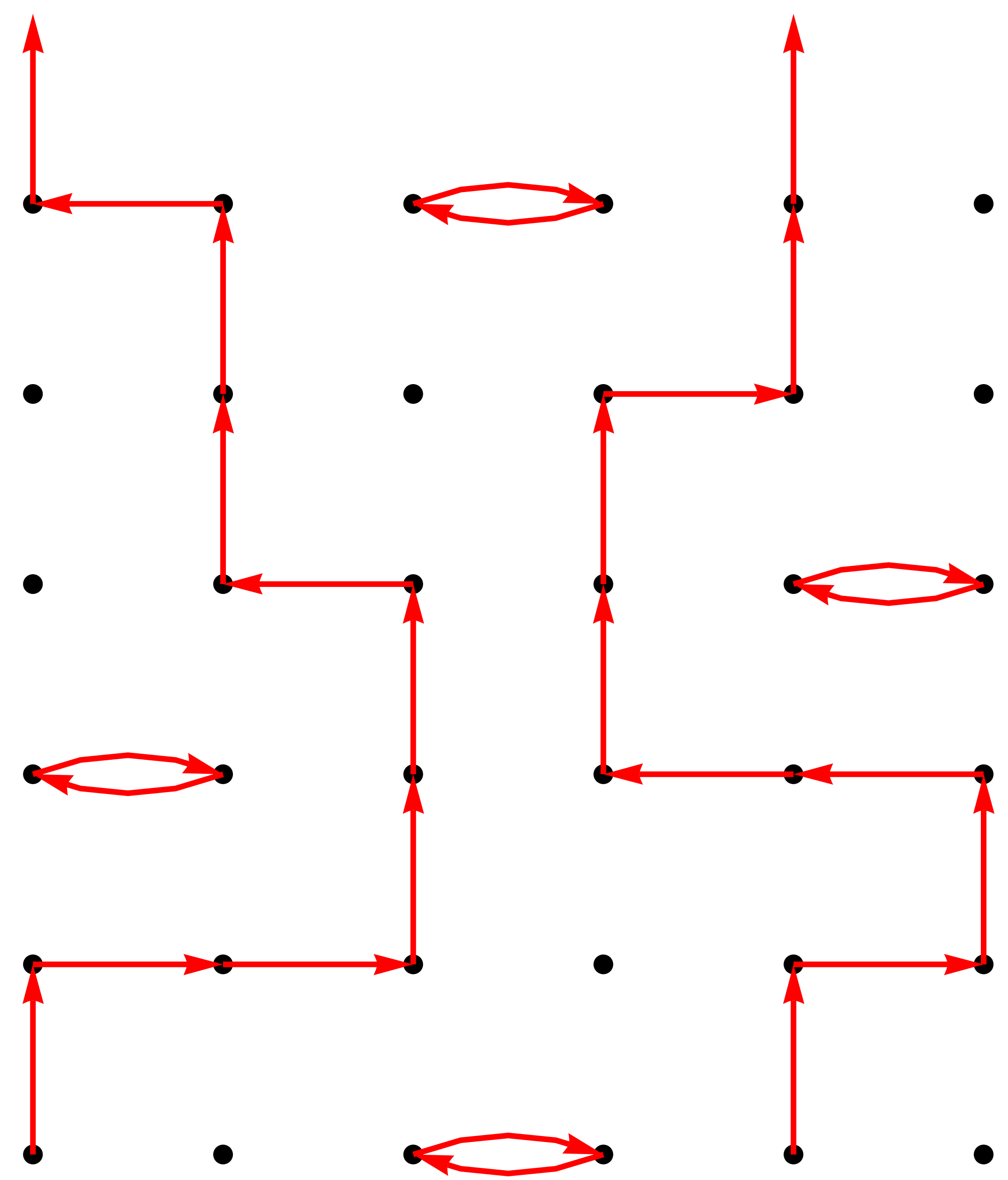
$$Z(\mu) = \int [d\psi^\dagger][d\psi] e^{-S(\mu)}$$

- Fermion paths form closed loops (due to Grassmann integration)
- Fermion loops are self-avoiding (Pauli exclusion)
- Single-hop fermion "bubbles" allowed



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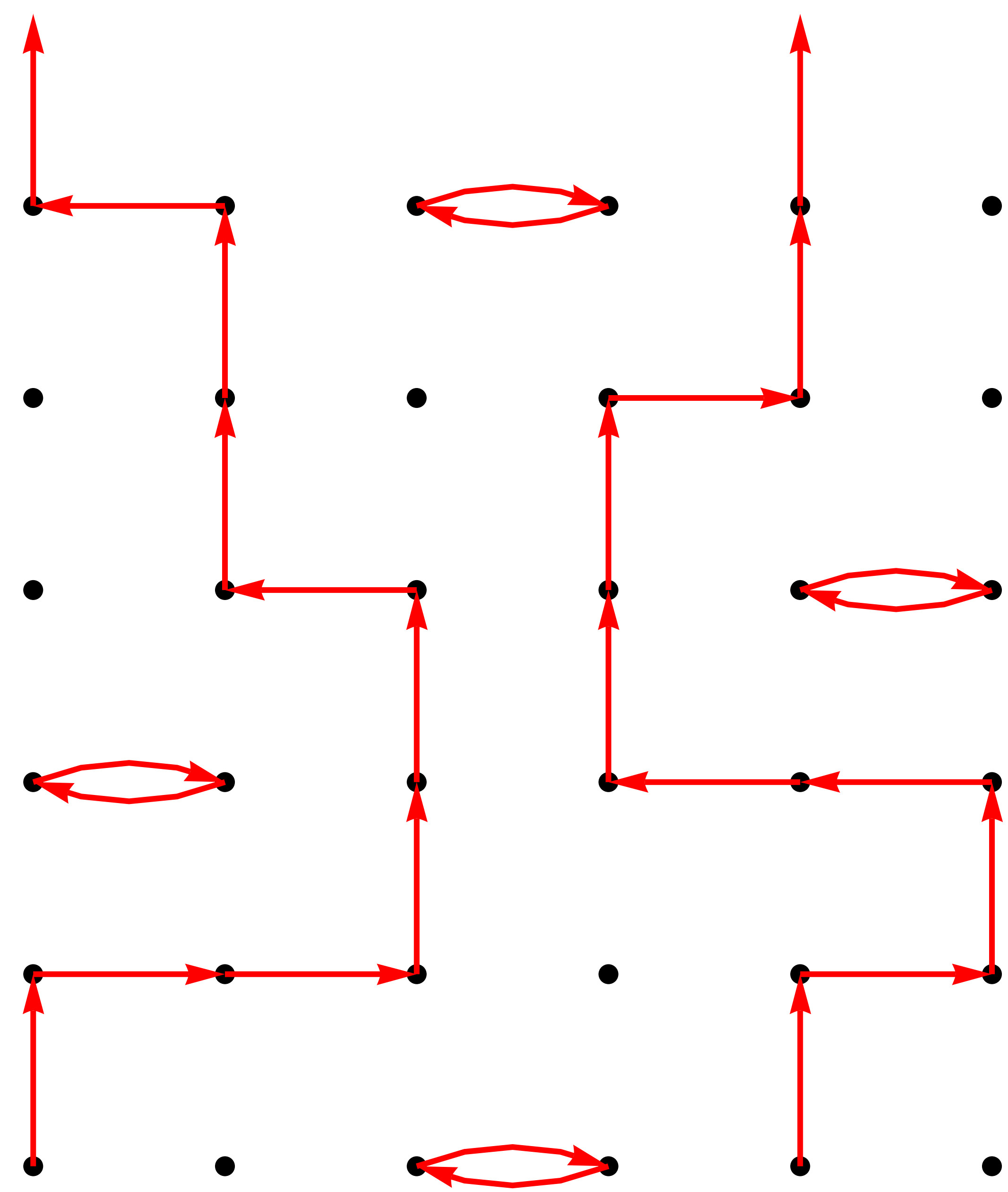
$$Z(\mu) = \int [d\psi^\dagger][d\psi] e^{-S(\mu)}$$

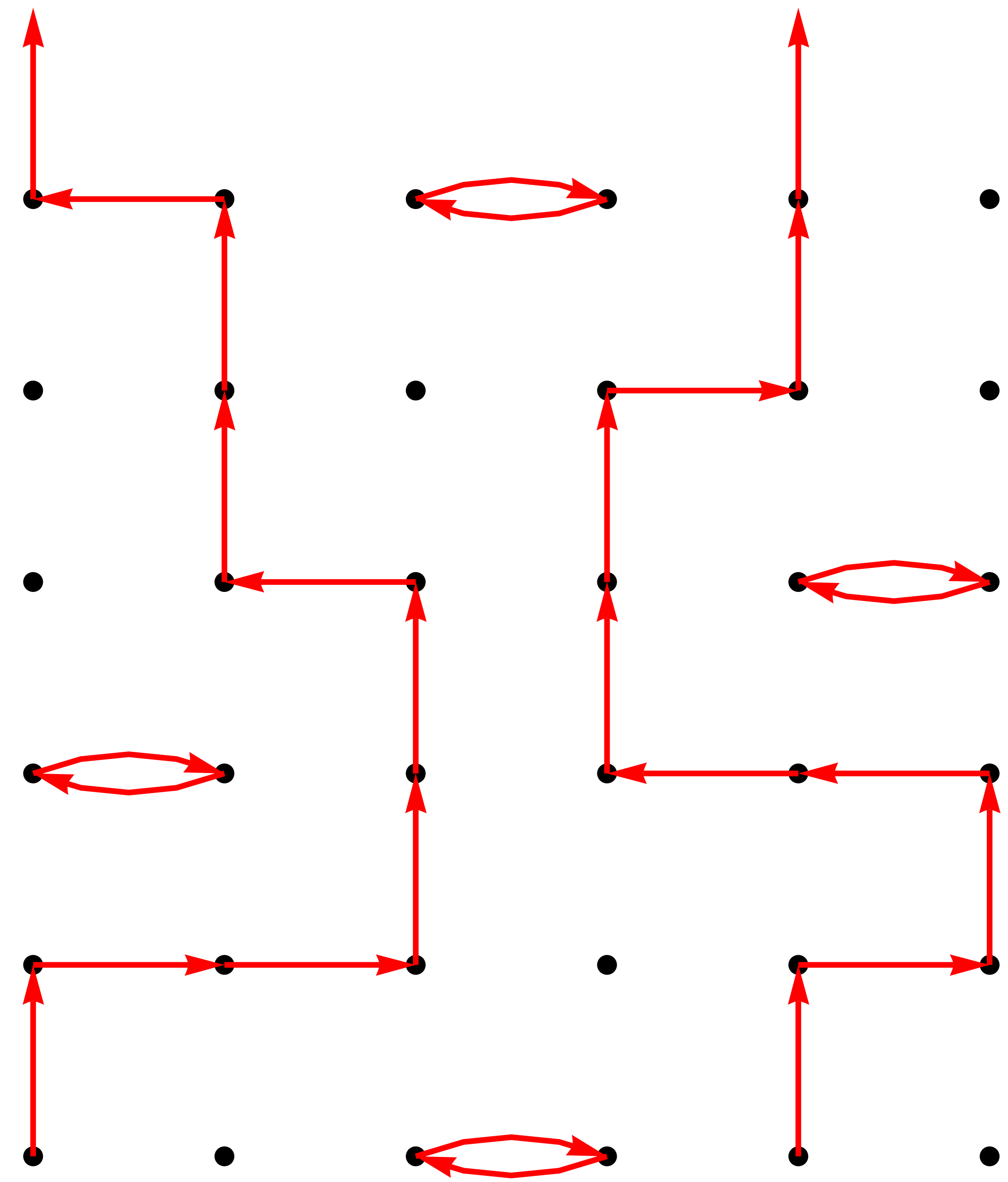
- All fermion loops come with a minus sign
- Every winding in time comes with a minus sign (APBCs)
- Fermion loops in time are disjoint (Pauli exclusion)



Consider a hopping parameter expansion

$$Z(\mu) = \sum_{\{c_\sigma\}\in C} \left[\prod_\sigma \left(1 + \frac{1}{m_\sigma}\right)^{\mathcal{S}(c_\sigma)} \left(\frac{1}{2m_\sigma}\right)^{\mathcal{B}_s(c_\sigma)} (-1)^{\mathcal{F}(c_\sigma)} e^{\mu_\sigma \mathcal{B}_\tau(c_\sigma)} \right]$$

- S = # unvisited sites
- B_s = # space-like links
- B_t = # time-like links
- F = # space-like fermion bubble loops



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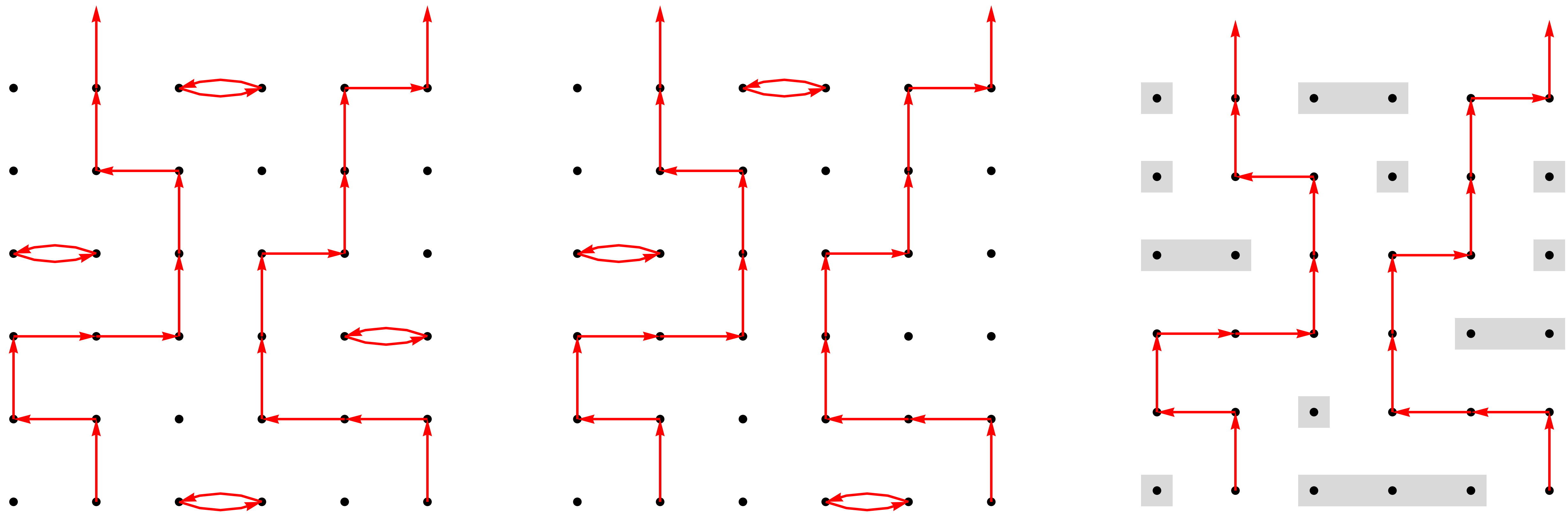
...but we still have a sign problem.

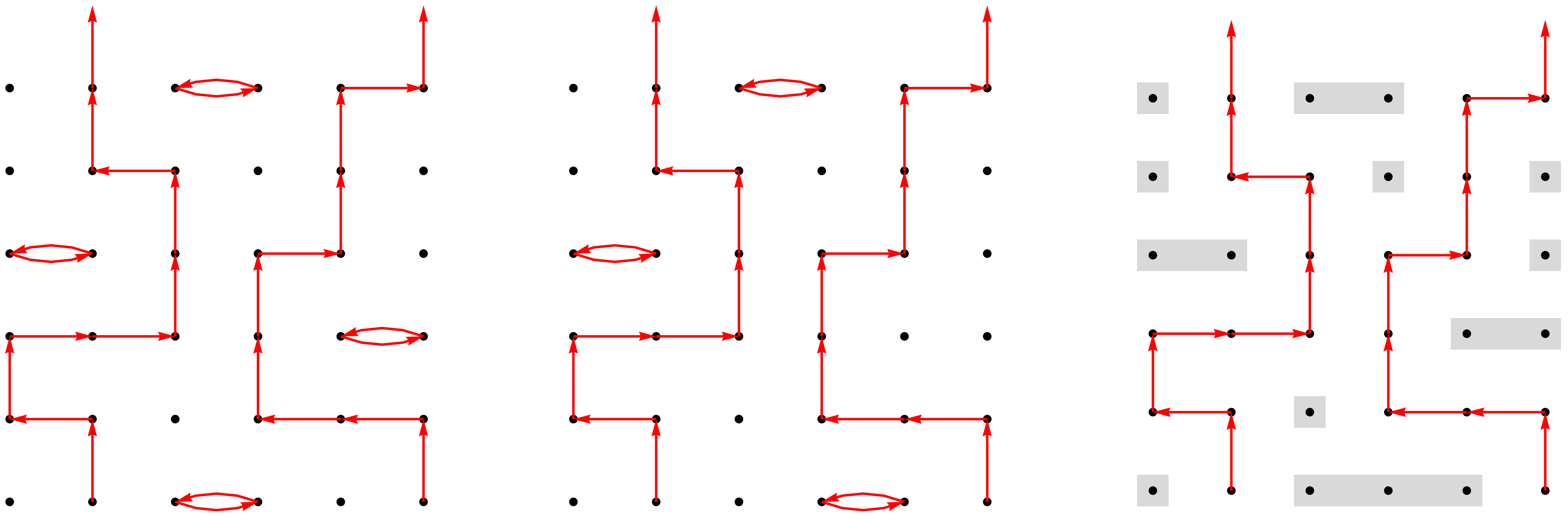
Solving the sign problem

Observation: configurations fall into classes designated by the time-like loops

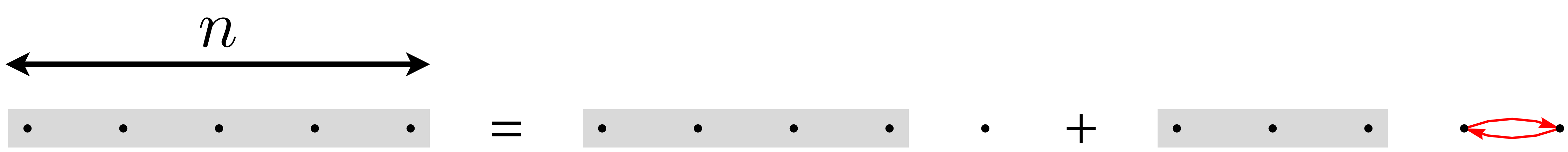


Solving the sign problem

Idea: sum all up the fermion bubbles within each class

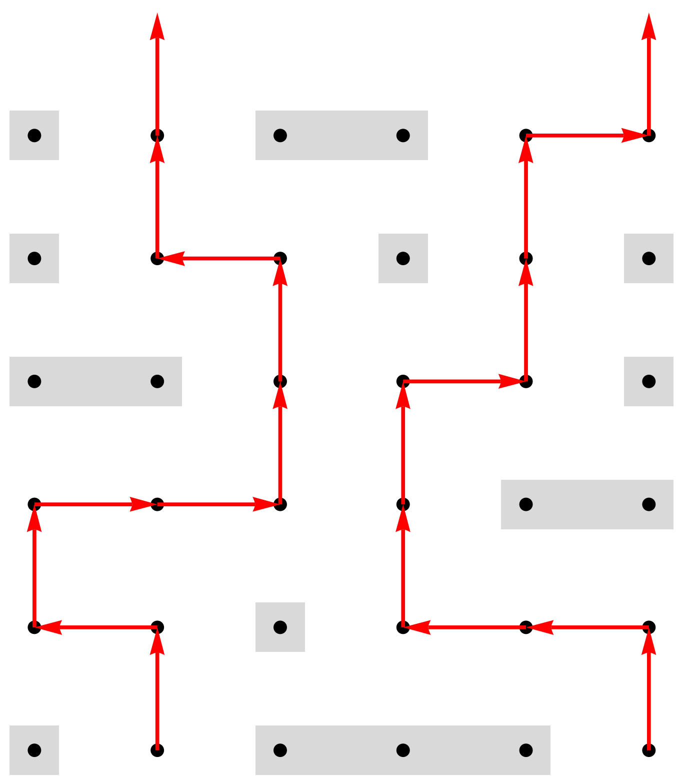


Evaluation of the fermion bubble sum



$$z_n(m) = z_{n-1}(m)\left(1 + \frac{1}{m}\right) - \frac{1}{(2m)^2} z_{n-2}(m)$$

$$z_0(m) = 1 \qquad z_1(m) = 1 + \frac{1}{m}$$



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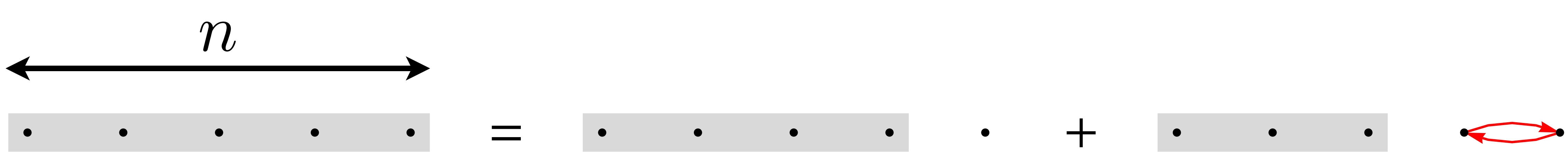
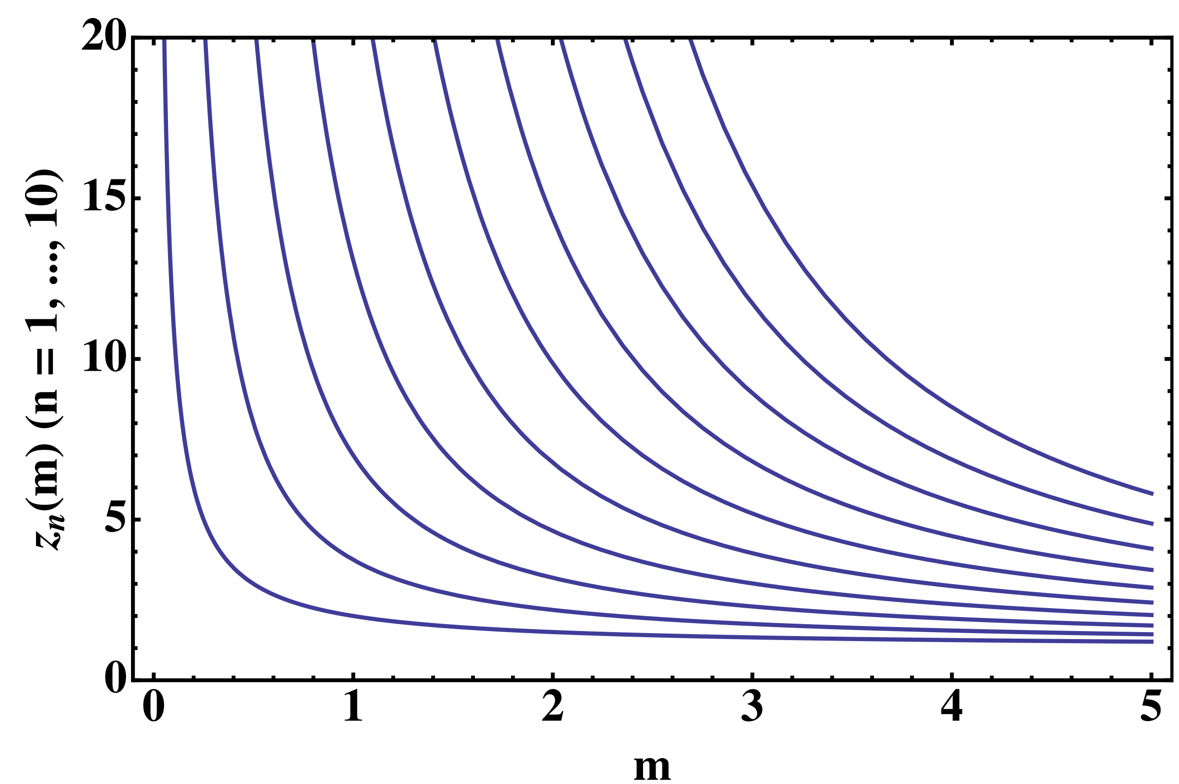


$$z_n(m) = \frac{1}{2^{n+1}\sqrt{1+2/m}}\left[\left(1+\frac{1}{m}+\sqrt{1+\frac{2}{m}}\right)^{n+1} - \left(1+\frac{1}{m}-\sqrt{1+\frac{2}{m}}\right)^{n+1}\right]$$



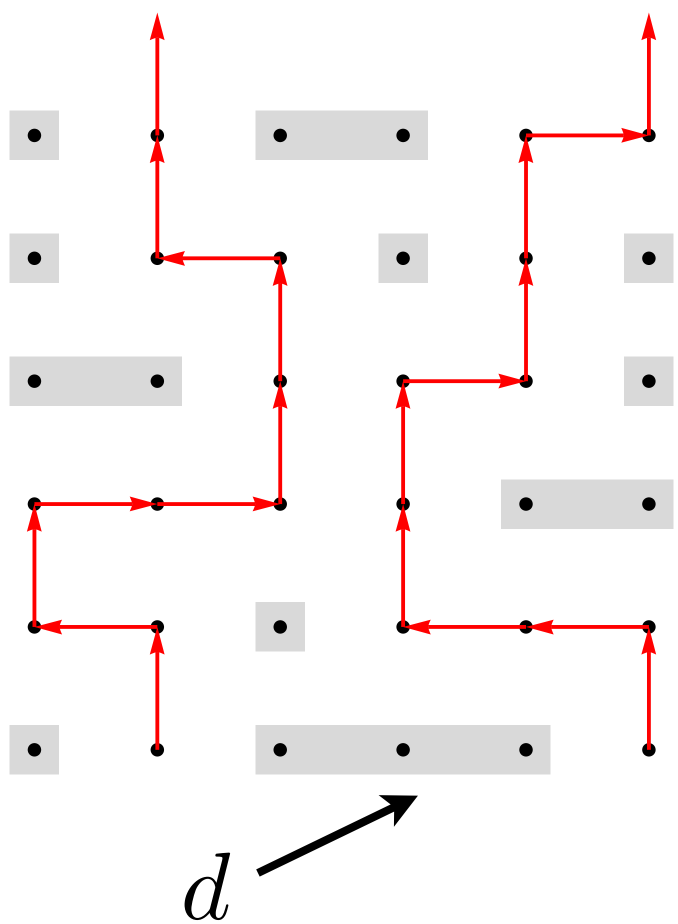
Final result for free fermions

$$Z(\mu) = \sum_{\{c_\sigma\}\in C^*} \left[\prod_\sigma \left(\prod_{d\in\mathcal{D}(c_\sigma)} z_{\mathcal{L}(d)}(m_\sigma) \right) \left(\frac{1}{2m_\sigma} \right)^{\mathcal{B}_s(c_\sigma)} e^{\mu_\sigma \mathcal{B}_\tau(c_\sigma)} \right]$$

$\mathcal{D}(c) =$ set of domains d associated with c

$\mathcal{L}(d) =$ length of d

C* = set of all possible
"bubble free" configurations



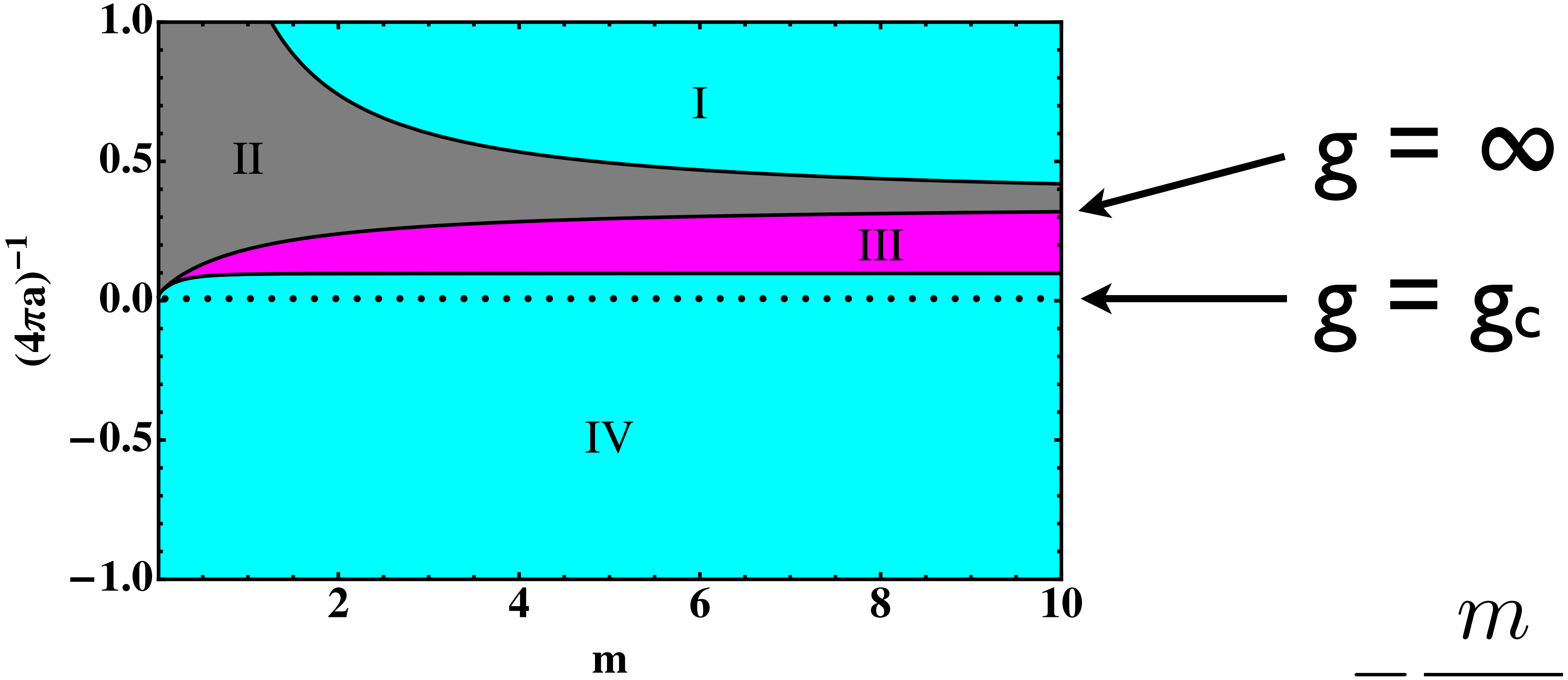
Final result for interacting fermions

$$Z(\mu) = \sum_{\{c_\sigma\}\in C^*} \left[\prod_\sigma \left(\prod_{d\in\mathcal{D}(c_\sigma)} z_{\mathcal{L}(d)}(m_\sigma) \right) \left(\frac{1}{2m_\sigma} \right)^{\mathcal{B}_s(c_\sigma)} e^{\mu_\sigma \mathcal{B}_\tau(c_\sigma)} \right] (1+g)^{\mathcal{B}_\tau(\cap_\sigma c_\sigma)}$$

$$c_a \cap c_b \cap c_c \cap c_d$$

- Resulting partition function free of sign problems even with:
  - mass imbalance
  - polarization imbalance
  - repulsive interactions (provided g > -1)

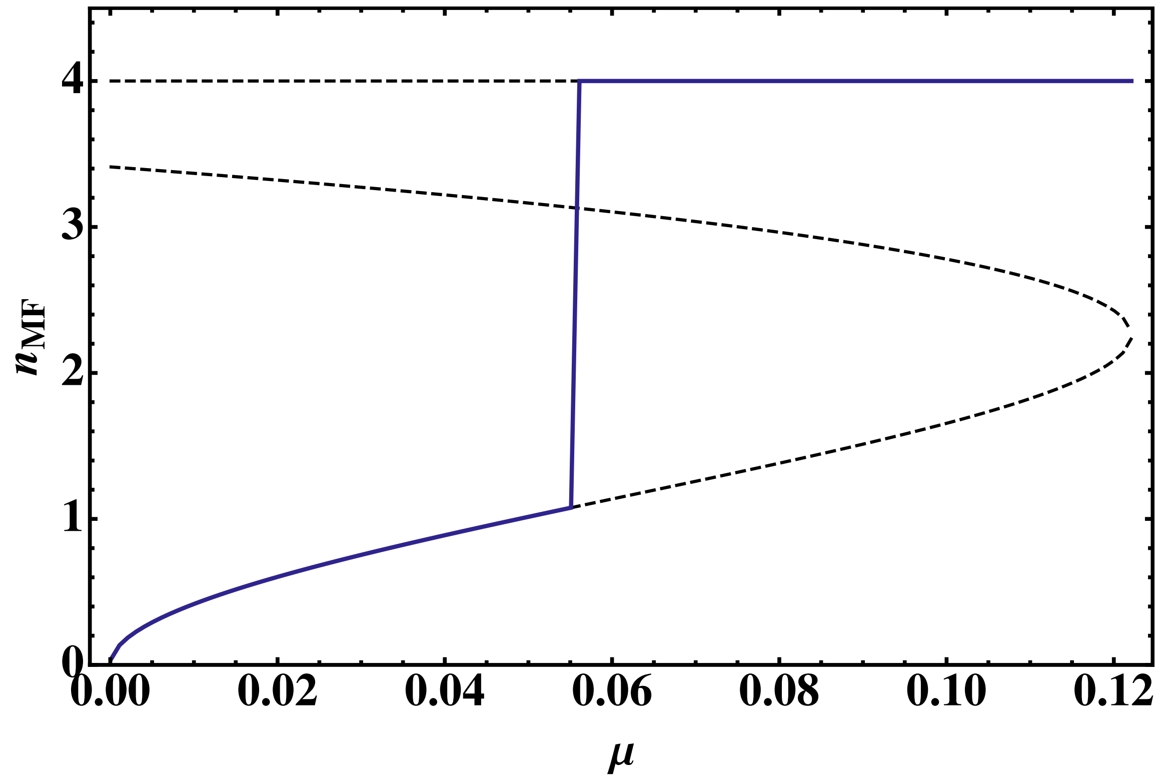
Mean-field results: m-a^{-1} density phase diagram



$$-\frac{m}{4\pi a} = \frac{1}{g} - \frac{1}{g_c}$$

- I & IV: zero density regime
- II: inaccessible within variational approach (complex action)
- III: saturated lattice, density equals four

Mean-field results: free energy & number density



Continuum exists within mean field analysis:

- $a \to \infty$
- $b_s \ll n^{-1} \ll L \ , \quad n = N/L$
- na held fixed

Summary/Outlook

- World-line representation for the partition function for non-relativistic fermions in on dimension that is completely free of sign problem, irrespective of:
    - mass imbalance
    - polarization imbalance
    - sign of interaction strength
- May be used to simulate a universal four-component Fermi gas, which exhibits many features in common with “unitary fermions” in three dimensions
    - Bertsch parameter, gap, contact, few-body energies in a trap, etc...

Summary/Outlook

- In progress: few-body studies in the canonical ensemble
  - energies of up to eight fermions in a harmonic trap
  - benchmark comparisons with known results of Nishida & Son: four and five fermion systems
- Implementation of worm algorithm (nearing completion)
- Ultimate goal: "continuous time" Monte Carlo simulations
  - eliminate extrapolation in temporal lattice spacing
  - presumably more efficient than discrete time Monte Carlo

Thank you!