New fermion discretizations and their applications

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Why New Fermions?

(1) Further understanding on lattice field theory

(2) Improvement of lattice QCD simulations
Lattice fermion improvement

**Wilson:** $O(a)$ errors & bad chiral properties

- **Smeared-link clover** (UV filtered, $O(a)$ improved)
  DeGrand, Hasenfratz, Kovacs, MILC(98), BMW’s intensive works
- **Twisted-mass** (unphysical zero-mode removed, $O(a)$ improved)
  Frezzotti, et al. ALPHA(00), ETM’s intensive works

**Staggered:** taste breaking at $O(a^2)$

- **HISQ** (fat-link & $O(a^2)$ Symanzik)
  Follana, et al.(06), MILC’s intensive works
- **HYP** Hasenfratz, Knechtli(01)
- **Fat7** Orginos, Toussaint et al.(99)
- **Asqtad** Lepage (98)

**Domain-wall, Overlap:** Numerical cost

- **Fixed topology** (kernel zero-mode removed, locality)
  Fukaya, et al. (06), JLQCD intensive works
- **Reweighting** (enlarge 5th size, chiral properties)
  Hasenfratz et al.(08), Ishikawa, et al. (10)
- **Hypercube overlap** (perfect kernel→locality, scaling)
  Bietenholtz, et al. (99)(12) etc....
New setups can contribute?

1. **Flavored mass**
   
   Adams (09)      cf.) Golterman, Smit (84)
   
   Staggered overlap $\rightarrow$ CPU time reduction (overlap)?
   Taste symmetry improved (staggered)?

2. **Central branch**
   
   Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)
   
   Wilson w/o additive renorm. $\rightarrow$ Chiral symmetry (No fine-tuning?)
   $O(a)$ improved?

3. **Minimal-doubling**
   
   Karsten(81) Wilczek(87) Creutz(07) Borici(07)
   
   Chiral two-flavor w/ ultra locality $\rightarrow$ Better chiral property?
1. Flavored mass
Naive

\[ C_\mu = (T_{+\mu} + T_{-\mu})/2 \]
\[ T_{\pm\mu} \psi_n = U_{n,\pm\mu} \psi_{n\pm\mu} \]

U(4)×U(4)

Wilson

\[ \sum_\mu C_\mu \]

\[ 1 \quad 4 \quad 6 \quad 4 \quad 1 \]

Re \( \lambda \)

Im \( \lambda \)
Throughout this paper, we consider the nondimensionalized action. As is discussed in [7, 42], introduce the spin-flavor representation, which simplifies the identification of symmetry in the

In this section, we first review the U(4) Wilson fermion with emphasis on the symmetry enhancement and its spontaneous breakdown.

the Bori¸ ci-Creutz minimally doubled fermion and discover that a similar type of symmetry is spontaneously broken by pion condensation. Finally, we explore the Karsten-Wilczek and however, that an additional U(1) vector symmetry is realized by tuning

the U(4) Wilson fermion via the spin-flavor representation. In section 3, we discuss the symmetries of the naive fermion in [7, 42] using the spin-flavor representation.

This paper is organized as follows. In section 2, we revisit the symmetries of the naive fermions with flavored mass terms

\[ \sum_{\mu} C_{\mu} \]

\[ C_{\mu} = (T_{+\mu} + T_{-\mu})/2 \]

\[ T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu} \]

U(4)×U(4) →

\[ \Gamma_X^{(+)} \in \left\{ 1_4, (-1)^{n_1+\ldots+n_4}\gamma_5, (-1)^{n_5}\gamma_{\mu}, (-1)^{n_4}i\gamma_{\mu}\gamma_5, (-1)^{n_{\mu,\nu}}[\gamma_{\mu}, \gamma_{\nu}]_2 \right\} \]

\[ \Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\ldots+n_4}1_4, \gamma_5, (-1)^{n_5}\gamma_{\mu}, (-1)^{n_{\mu,\nu}}[\gamma_{\mu}, \gamma_{\nu}]_2 \right\} \]

\[ \psi_n \rightarrow \psi'_n = \exp \left[ i \sum_{x} \left( \theta_x^{(+)}\Gamma_x^{(+)} + \theta_x^{(-)}\Gamma_x^{(-)} \right) \right] \psi_n , \quad \bar{\psi}_n \rightarrow \bar{\psi}'_n = \bar{\psi}_n \exp \left[ i \sum_{x} \left( -\theta_x^{(+)}\Gamma_x^{(+)} + \theta_x^{(-)}\Gamma_x^{(-)} \right) \right] \]
Construction

Then add mass (i.e., shift spectrum) to make it commute with the following anti-linear operation constraint invariant under the transformations with arbitrary complex potential. More precisely, in weak-coupling limit, two of 16 species have zero imaginary potential. The massless action is given by

\[ \mathcal{S} = \gamma_{\mu} \gamma_{\nu} \psi \bar{\psi} U_{n,\mu} U_{n,\nu} \psi \bar{\psi} \]

from the strong coupling analysis. It obviously breaks the hypercubic symmetry into the spatial cubic symmetry. It also breaks C, P and T symmetries into CT and P symmetry. We line up symmetries of this viewpoint of the universality class, these two theories belong to the same class. It is equivalent since the gamma matrices are differently defined between them as

\[ \psi_{\nu} = \psi_{\nu}^0 + \sum_{\mu = 1}^{16} C_\mu \psi_{\nu,\mu} \]

and ˇ\bar{\psi}_n = U_n,\mu \psi_n,\mu \bar{\psi}_n\mu.

Employing the spin-flavor representation, we investigate the structures of the doubler-mixing formulation. We first revisit the U(4) symmetries of the naive fermion is broken by flavored-mass terms.

Naive

Wilson

\[ \sum_\mu C_\mu \]

\[ C_\mu = (T_{\mu+} + T_{\mu-})/2 \]

\[ T_{\mu\nu} \psi_n = U_{n,\mu} \psi_{n,\nu} \]
Then add mass (ie. shift spectrum) to make

Include taste-dependent mass term:

Naive

\[ \sum \text{sym.} \ C_1 C_2 C_3 C_4 \]

\[ C_\mu = (T_{+\mu} + T_{-\mu})/2 \]

\[ T_{\pm \mu} \psi_n = U_{n,\pm \mu} \psi_{n\pm \mu} \]

Wilson’

\[ \Gamma^{(\pm)}_X \in \left\{ 1_4, (-1)^{n_1+\ldots+n_4} \gamma_5, \right\} \]

\[ (-1)^{n_{\mu,\nu}} \frac{[\gamma_\mu, \gamma_\nu]}{2} \]

U(4)×U(4) → U(2)×U(2)

Kimura (10)

TM, Lehner (12)
**Naive flavored mass**

\[ M_V = \sum_{\mu} C_\mu, \quad \text{Vector (1-link)} \]

\[ M_T = \sum_{\text{perm. sym.}} \sum_{\nu} C_\mu C_\nu, \quad \text{Tensor (2-link)} \]

\[ M_A = \sum_{\text{perm. sym.}} \prod_{\nu} C_\nu, \quad \text{Axial-V (3-link)} \]

\[ M_P = \sum \prod_{\text{sym. } \mu=1}^{4} C_\mu, \quad \text{Pseudo-S (4-link)} \]

- gamma-5 hermiticity
- 2nd derivative terms
  \[ \sum_n \bar{\psi}_n (M_P - 1) \psi_n \rightarrow -a \int d^4 x \bar{\psi}(x) D_\mu^2 \psi(x) + O(a^2) \]
- Cousins of Wilson fermion
Naive $\Rightarrow$ Flavored-mass $\Rightarrow$ Wilson$^{(o)}$ $\Rightarrow$ Domain-wall Overlap$^{(o)}$
Naive → Staggered
\(\text{Flavored-mass}\)

\(\text{Spin diag.}\) → \(\text{St. Wilson}\)

\(\text{Flavored-mass}\)

\(\text{Wilson}^{(\cdot)}\) → \(\text{Domain-wall}^{(\cdot)}\)
\(\text{Overlap}^{(\cdot)}\)

\(\text{Spin diag.}\) → \(\text{St. Wilson}\) → \(\text{St. Dm-wall}\)

\(\text{St. Overlap}\)

\(\text{Faster domain-wall & overlap} \, ??\)

Staggered

\[ \{C_0, \Xi_\mu, I_s, R_{\mu\nu}\} \times \{U^\epsilon(1)\}_{m=0} \]

St. Wilson  Adams (09)

\[ \{C_0, \Xi'_\mu, R_{\mu\nu}\} \]

- Practical form

\[ \eta_\mu D_\mu + r(1 + M_A) + m \]

Wilson-like term  mass parameter

\[ M_A = \epsilon_x \sum_{\text{sym.}} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4 \]

With this mass shift  \( \xi_5=-1 \rightarrow \text{physical sector} : \ell \)

\( \xi_5=+1 \rightarrow \text{decoupled sector} : \hat{h} \)

\[ \epsilon_x \sum_{\text{sym.}} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4 \sim (1 \otimes \xi_5) + O(a) \]

Staggered

\[ \{ C_0, \Xi_\mu, I_s, R_{\mu\nu} \} \times \{ U^\epsilon(1) \}_{m=0} \]

St. Wilson Adams (09)

No parameter tuning for Lorentz symmetry!

- Practical form

\[ \eta_\mu D_\mu + r(1 + M_A) + m \]

Wilson-like term mass parameter

\[ M_A = \epsilon_x \sum_{\text{sym.}} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4 \]

With this mass shift \( \xi_5 = -1 \rightarrow \) physical sector: \( \ell \)

\( \xi_5 = +1 \rightarrow \) decoupled sector: \( \hbar \)
**Staggered-Wilson (Domain-wall, Overlap)**

How to apply : As Wilson → Mass parameter tuning required  
As Domain-wall → 5th dimension introduced  
As Overlap → Overlap formula with StWil kernel

- Spin diagonalization  Creutz, Kimura, TM (10)

\[
\bar{\psi}_x C_1 C_2 C_3 C_4 \psi_x \rightarrow \pm \chi_x (\epsilon \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4) \chi_x
\]

- Index theorem  Adams (09)  Follana, Azcoiti, Di Carlo, Vaquero (11)

\[
H_{SW}(m) = \epsilon (D_{SW} - m)
\]

\[
\lambda(m)
\]

\[
\text{Index}(D_{sw}) = - \text{Spectral flow}(H_{sw})
\]

\[
H_W(m) = \gamma_5 (D_W - m)
\]

cf.) Edwards, Heller, Narayanan (98)
• **Aoki phase**  Creutz, Kimura, TM(11) TM, Nakano, Kimura, Ohnishi(12)

Strong-coupling lattice QCD & 2d model  
→ Implies parity-flavor broken phase

*ChPT analysis required → 1st or 2nd order?*

• **Another type (Hoelbling type)**  Hoelbling (10), de Forcrand, Kurkela, Panero (10)

\[
\sum_{\mu \nu=12,34} i \eta_{\mu \nu} \eta_{\mu} \epsilon_{\mu \nu} (C_\mu C_\nu + C_\nu C_\mu) \\
\sim (1 \otimes i (\sigma_{12} + \sigma_{34})) + O(a)
\]

→ Requires fine-tuning of parameters for Lorentz sym. continuum  Sharpe (12)

*Let’s focus only on Adams type.*
§ Potential problems of $\eta_\mu D_\mu + r(1 + M_A)$

1. Lorentz symmetry restored?

Euclidian Lorentz symmetry, C, P, T

$\rightarrow$ likely to be restored from $\{C_0, \Xi_\mu, R_{\mu\nu}\}$ No parameter tuning!

2. Multi-link terms require numerical costs?

(i) 24 terms for symmetric sum, (ii) 4 transporters

VS

One component fermion (small matrix size)
§ Potential advantages of $\eta_{\mu}D_{\mu} + r(1 + M_A)$

1. could reduce numerical costs in 2-flavor overlap
   
   One-component action $\rightarrow$ Small matrix size of propagator

2. could reduce influence of taste-breaking for 2-flavor
   
   Staggered sym. $\quad$ vs $\quad$ 4 tastes

   Halved staggered sym. $\quad$ vs $\quad$ 2 tastes

The situation should be different, but better or worse?
1. Numerical costs reduced?

Staggered-Overlap Dirac propagator

- Small matrix size
  Requires fewer Matrix-Vector multiplications for sign function!

- 4-link hopping terms
  Gauge fluctuation is raised to 4th power!
  → splitting of two branches reduced

Staggered-Wilson is better as an overlap kernel, but not much better.

CG solver with MxV (12^4, m=0.1)

![Graph showing comparison between different overlap kernels and CG iterations.](image)
2. How about taste breaking? Sharpe (12)

**Pion spectrum**

§ Staggered \{C_0, \Xi_\mu, I_s, R_{\mu\nu}\} \times \{U^c(1)\}_{m=0} \rightarrow \{C_0, \Xi_j, I_s, R_{ij}\}

- **Transfer-matrix sym.**

- **classify 15 pseudoscalar operators** Golterman (1986)

  1 : \xi_4, \xi_{45}, \xi_5, 7 irreps

  3 : \xi_i, \xi_{i5}, \xi_{ij} \xi_{i4}

§ Staggered-Wilson \{C_0, \Xi'_\mu, R_{\mu\nu}\} \rightarrow \{C_0, \Xi'_j, R_{ij}\}

- **Irreps mix in \xi_5 pairs**

- **Physical sector**

  \(\bar{\ell}(\gamma_5 \otimes 1)\ell\) \(\eta'\)

- **States in 3d irrep**

  \(\pi_0, \pi^\pm\)

Discrete symmetries are sufficient for degenerate pion triplet!
◆ Short summary

• Adams fermion will work as 2-flavor Wilson.

• Taste-breaking exists, but small enough to have degenerate pion triplet.

• Further study is needed to reveal numerical merit or demerit.

How about other mesons and baryons?

Usual improvement works? (Fixed topology, smearing)
Naive \quad \rightarrow \quad Wilson^{(s)} \quad \rightarrow \quad Domain-wall^{(s)}

Staggered \quad \rightarrow \quad St.
Wilson \quad \rightarrow \quad St.
Dm-wall

\textit{Flavored-mass}

Spin diag.
2. Central-branch
2. Central-branch

- Wilson w/o onsite term \( M_W \equiv m + 4r = 0 \)

\[
S = \frac{1}{2} \sum_{x, \mu} \bar{\psi}_x \left[ \gamma_{\mu} (U_{x, \mu} \psi_{x+\mu} - U_{x,-\mu} \psi_{x-\mu}) - (U_{x, \mu} \psi_{x+\mu} + U_{x,-\mu} \psi_{x-\mu}) \right]
\]

\( \Rightarrow \) Another \( U(1) \)!

\[
\psi_x \to e^{i\theta (-1)^{x_1+x_2+x_3+x_4}} \psi_x, \quad \bar{\psi}_x \to \bar{\psi}_x e^{i\theta (-1)^{x_1+x_2+x_3+x_4}}
\]

\[
\Gamma_X^{(+)} \in \left\{ 1_4, (-1)^{n_1+\cdots+n_4} \gamma_5, (-1)^{n_\mu} \gamma_\mu, (-1)^{n_\mu} i \gamma_\mu \gamma_5, (-1)^{n_\mu, \nu} \frac{[\gamma_\mu, \gamma_\nu]}{2} \right\}
\]

\[
\Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\cdots+n_4} 1_4, \gamma_5, (-1)^{n_\mu} \gamma_\mu, (-1)^{n_\mu} \gamma_\mu \gamma_5, (-1)^{n_\mu, \nu} \frac{[\gamma_\mu, \gamma_\nu]}{2} \right\}
\]
2. Central-branch

- Wilson w/o onsite term \( M_W \equiv m + 4r = 0 \)

\[
S = \frac{1}{2} \sum_{x,\mu} \bar{\psi}_x \left[ \gamma_\mu (U_{x,\mu} \psi_{x+\mu} - U_{x,-\mu} \psi_{x-\mu}) - (U_{x,\mu} \psi_{x+\mu} + U_{x,-\mu} \psi_{x-\mu}) \right]
\]

\[ \Rightarrow \text{Another U(1)!} \]

\[ \psi_x \rightarrow e^{i\theta(-1)^{x_1+x_2+x_3+x_4}} \psi_x, \quad \bar{\psi}_x \rightarrow \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}} \]

\( \Gamma_X^{(+)} \in \left\{ 1_4, \right\} \)

\( \Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\ldots+n_4} 1_4, \right\} \)

\( \gamma_5 \otimes \xi_5 \) Prohibits additive mass renormalization!

SSB gives NG boson!
**Strong-coupling QCD**  
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

\[
cosh(m_{SPA}) = 1 + \frac{2M_W^2(16 + M_W^2)}{16 - 15M_W^2}
\]

NG boson associated with SSB of U(1)

- Pion (eta) condensate \( \langle \psi \gamma_5 \psi \rangle \neq 0 \)
- No chiral condensate \( \langle \bar{\psi} \psi \rangle = 0 \)

§ Advantages

- No additive mass renormalization (no fine-tuning)
- SSB of U(1) and massless NG boson
- No O(a) errors

§ Potential drawbacks

- Sign problem
- U(1) problem
- Quark mass

Could be a new possibility of 12-flavor lattice QCD
Central points for other flavored masses

• For other naive flavored mass terms

\[ M_A : U(1) \text{ restored} \]
\[ M_T : U(2) \text{ restored} \]
\[ M_P : U(4) \text{ restored} \]

• For staggered flavored mass terms

\[ M_{\mathcal{A}} : C_T \Xi, C_T'I \text{ restored} \]
\[ M_{\mathcal{H}} : C_T' \text{ restored} \]

\[ C_T' : \chi_x \rightarrow \bar{\chi}_x^T, \bar{\chi}_x \rightarrow \chi_x^T, U_{x,\mu} \rightarrow U_{x,\mu}^* \]

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3. Minimal-doubling
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Flavored imaginary chemical potential term lifts species degeneracy.
cf.) Flavored mass in Wilson

\[
\sum_\mu (1 - \cos p_\mu)
\]

\[
(i) \gamma_4 \sum_{j=1}^{3} (1 - \cos p_j) \rightarrow \text{keeping one chiral sym.}
\]

Finite-mass system(Wil) $\Leftrightarrow$ Finite-density system(FCP)

◆ Advantage
  - U(1) chiral symmetry
  - Ultra-local
  - 2 flavor possible

◆ Drawbacks
  - Hypercubic symmetry breaking
  - Tuning parameters for a correct continuum limit

Bedaque, Buchoff, Tiburzi, Walker-Loud(08)
Capitani, Creutz, Weber, Wittig (09)(10)
**Symmetries**

Bedaque, Buchoff, Tiburzi, Walker-Loud (08)

1. $U(1)$ chiral symmetry
2. P
3. CT

**Counterterms**

Capitani, Creutz, Weber, Wittig (09)(10)

$$\dim 3 \quad \bar{\psi}_n i \gamma_4 \psi_n \quad \dim 4 \quad \bar{\psi}_n \gamma_4 D \psi_n \quad F_{i4} F_{i4}$$

**Fine-tuning of three parameters are required for Lorentz sym.**

**Chiral phase structure**

TM (12)

*Nontrivial phase diagram in the parameter space*
Finite $(T, \mu)$ QCD with FCP

Still fine-tuning for $O(1/a)$ chemical potential renorm.... cf.)additive mass in Wilson

But the discrete symmetries suit this case.

P.Hasenfratz, Karsch (83)

\[
S_{md} = \sum_x \left[ \frac{1}{2} \sum_{j=1}^{3} \bar{\psi}_x \gamma_j (U_{x,x+j}\psi_{x+j} - U_{x,x-j}\psi_{x-j}) + \frac{1}{2} \bar{\psi}_x \gamma_4 \left( e^\mu U_{x,x+4}\psi_{x+4} - e^{-\mu} U_{x,x-4}\psi_{x-4} \right) 
+ \frac{i}{2} \sum_{j=1}^{3} \bar{\psi}_x \gamma_4 (2\psi_{x} - U_{x,x+j}\psi_{x+j} - U_{x,x-j}\psi_{x-j}) + id_3 \bar{\psi}_x \gamma_4 \psi_x \right]
\]

§ Strong-coupling study

Effective potential of $\sigma$ as a function of $T, \mu$ and $d_3$

Chiral phase structure

- 1st and 2nd phase transition ($m=0$)
- 1st, critical point and crossover ($m\neq 0$)

New possibility of $(T,\mu)$ lattice QCD!
4. **Summary**

1. **Flavored-mass terms** give us new types of Wilson and overlap fermions.

2. **Staggered-Wilson** can be an alternative Wilson and overlap for 2-flavor QCD (3 degenerate pion spectrum)

3. **Central-branch fermion** is a new possibility of use of Wilson for many-flavor QCD without fine-tuning of parameters.

4. **Flavored-chemical-potential fermion** would be useful for finite-temperature & density lattice QCD.
Related talks

Tuesday 15:30 Room 8      Taro KIMURA
“QCD Phase diagram with 2-flavor discretization”

Wednesday 9:30 Room 5    Takashi NAKANO
“Strong coupling analysis of Aoki phase in St-Wil fermions”
Back-up slides
◆ **Spectral flow**

(i) Hermitian operator

\[
H(m) = \gamma_5 (D - m) \quad (H^2 = D^\dagger D + m^2 \geq 0)
\]

(ii) Eigenvalue flow \( \lambda_i(m) \)

\[
\lambda_0(m) = \mp m \quad \text{only for zero modes}
\]

zero mode : low-lying crossing

chirality : minus the sign of slope

![Graph showing spectral flow](image)


• **lattice theory** (Wilson fermion)

(i) Hermitian operator

\[
H_W(m) = \gamma_5 (D_W - m)
\]

(ii) Eigenvalue flow

would-be zero modes : low-lying real crossing

approximate chirality : \( \lambda'(m) = -\psi(m)^\dagger \gamma_5 \psi(m) \)

**Index** \( (D_W) = - \) Spectral flow \( (H_W) \)

\[
\text{Index}(D_W) = (-1)^{d/2} Q
\]

※ Spectral flow :

Crossings counted with \( \pm \) slopes
For generalized Wilson fermions

\[ H_{gw} = \gamma_5 (D_{nf} - M_P) \]

\[ \text{Index}(D_{gw}) = -\text{Spectral flow}(H_{gw}) \]

\[ \text{Index}(D_{gw}) = 2^d (-1)^{d/2} Q \]

※ gauge configuration:

\[ U_{x,1} = e^{i \omega x_2}, \quad U_{x,2} = \begin{cases} 1 & (x_2 = 1, 2, \cdots, L - 1) \\ e^{i \omega L x_1} & (x_2 = L) \end{cases}, \quad \omega = 2\pi Q. \]

For staggered-Wilson fermions

\[ H_{sw} = \epsilon (D_{st} - M_f^{(A)}) = \Gamma_{55} (D_{st} - M_f^{(A)}) \]

\[ \text{Index}(D_{sw}) = -\text{Spectral flow}(H_{sw}) \]

\[ \text{Index}(D_{sw}) = 2^{d/2} (-1)^{d/2} Q \]

Index theorem holds for them.
**Overlap formulation**

negative-mass mode in $D_w \rightarrow$ massless mode in $D_{ov}$

> Low-lying crossings are far from high-lying ones

- **Generalized overlap**
  
  $$D_{go} = 1 + \gamma_5 \frac{H_{gw}(m)}{\sqrt{H_{gw}^2(m)}}$$

  *Any-flavor (1~15) overlap is possible!*

  cf.) 2 or 3-flavor overlap → lattice QCD
  12-flavor overlap → conformal window

- **Staggered-overlap**
  
  $$D_{so} = 1 + \Gamma_{55} \frac{H_{sw}(m)}{\sqrt{H_{sw}^2(m)}}$$

  *Less expensive overlap!*

  cf.) 1/4 matrix size → less CPU cost for Lanczos process
- **Shift symmetry** → broken to 2-link shift for $S_A$
  broken to 4-link shift for $S_H$

  $S_\rho : \chi_x \rightarrow \zeta_\rho(x)\chi_{x+\hat{\rho}}, \quad \bar{\chi}_x \rightarrow \zeta_\rho(x)\bar{\chi}_{x+\hat{\rho}}, \quad U_{\mu,x} \rightarrow U_{\mu,x+\hat{\rho}}$

  $S_\mu : \phi(p) \rightarrow \exp(ip_\mu)\Xi_\mu \phi(p)$

- **Axis reversal** → broken to shifted axis reversal

  $\mathcal{I}_\rho : \chi_x \rightarrow (-1)^x\rho\chi_{Ix}, \quad \bar{\chi}_x \rightarrow (-1)^x\rho\bar{\chi}_{Ix}, \quad U_{\mu,x} \rightarrow U_{\mu,Ix}$

  $\mathcal{I}_\rho : \phi(p) \rightarrow \Gamma_\rho \Gamma_5\Xi_\rho \Xi_5 \phi(Ip)$

- **Rotation** → remain in $S_A$
  broken to subgroup in $S_H$

  $\mathcal{R}_{\rho\sigma} : \chi_x \rightarrow S_R(R^{-1}x)\chi_{R^{-1}x}, \quad \bar{\chi}_x \rightarrow S_R(R^{-1}x)\bar{\chi}_{R^{-1}x}, \quad U_{\mu,x} \rightarrow U_{\mu,Rx}$

  $\mathcal{R}_{\rho\sigma} : \phi(p) \rightarrow \exp\left(\frac{\pi}{4}\Gamma_\rho \Gamma_\sigma\right)\exp\left(\frac{\pi}{4}\Xi_\rho \Xi_\sigma\right)\phi(R^{-1}p)$

- **Conjugation** → remain in $S_A$
  broken in $S_H$

  $\mathcal{C} : \chi_x \rightarrow \epsilon_x\bar{\chi}_x^T, \quad \bar{\chi}_x \rightarrow -\epsilon_x\bar{\chi}_x^T, \quad U_{\mu,x} \rightarrow U_{\mu,x}^*$

  $\mathcal{C} : \phi(p) \rightarrow \tilde{\phi}(-p)^T$

**Axis and Rotation** → $(\Gamma_4 \times| SW_{4,\text{diag}})$
Details of StWil symmetries

\[ \{ \Xi_\mu, I_s, R_{\mu\nu} \} \rightarrow \Gamma_4 \rtimes SW_4 \]

\[ \{ \Xi'_\mu, R_{\mu\nu} \} \rightarrow \Gamma_3 \rtimes SW_4 \]

Physical-sector symmetry

\[ \Xi'_j \Xi'_4 R^2_{j4} = \Xi_j \Xi_4 \sim (1 \otimes \sigma_j) \]

\[ \Xi'_4 R^2_{34} R^2_{12} = \Xi_4 I_s \sim (\gamma_4 \otimes 1) \]

\[ C_0 \Xi'_2 \Xi'_4 R^2_{24} \sim C \]
Details of timeslice symmetries

Enlarged staggered sym : \( \{C_0, \Xi_\mu, I_s, R_{\mu\nu}, T^{1/2}_\mu\} \) \quad \Xi^2_\mu = 1

\[ \rightarrow T^{1/2}_\mu \times \{C_0, \Xi_\mu\} \times \{R_{\mu\nu}, I_s\} = (\otimes_j Z_{N_\mu}) \times [\Gamma_{4,1} \times W_4] \]

Timeslice sym : \( T^{1/2}_\mu \times \{C_0, \Xi_\mu\} \times \{R_{ij}, I_s\} = (\otimes_j Z_{N_j}) \times [\Gamma_{4,1} \times W_3] \)

Relevant group at rest

\[ \Gamma_{4,1} \times W_3 \sim \{\{R_{ij}, \Xi_{ij}\} \times \{C_0, \Xi_4, \Xi_{123}, I_s\}\}/Z_2 \]
\[ = \{\{R_{ij}, \tilde{\Xi}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi_{kj}\} \times \{C_0, \Xi_4, \Xi_{123}, C_0 \Xi_4 I_s\}\}/Z_2 \]
\[ = [SW_4 \times \Gamma_{2,2}]/Z_2 \]

Staggered-Wilson

\( \{C_0, \Xi'_\mu, R_{\mu\nu}, T^{1/2}_\mu\} \sim \{\{R_{ij}, \Xi'_{ij}\} \times \{C_0, \Xi'_4, \Xi'_{123}, I_s\}\}/Z_2 \)
\[ = \{\{R_{ij}, \tilde{\Xi}'_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi'_{kj}\} \times \{C_0, \Xi'_4, \Xi'_{123}\}/Z_2 \]
\[ = [SW_4 \times \Gamma_{1,2}]/Z_2 \]
Dim3, 4: \( Q(1 \otimes \xi_F)Q \quad \bar{Q}(\gamma_\mu \otimes \xi_F)D_\mu Q \) for \( \xi_F = 1 \) or \( \xi_5 \) → \( \bar{\ell} \gamma_\mu D_\mu \ell \), \( \bar{\ell} \ell \)

Dim5 \( O(a) \): \( \bar{Q}(i\sigma_{\mu\nu}F_{\mu\nu} \otimes \xi_F)Q \) for \( \xi_F = 1 \) or \( \xi_5 \) → \( \bar{\ell}i\sigma_{\mu\nu}F_{\mu\nu}\ell \)

*No unphysical term nor taste-breaking term up to \( O(a) \)*

Dim6 \( O(a^2) \): 2 types of four-fermi operators \( \mathcal{L}_6^{FF(A)} \) and \( \mathcal{L}_6^{FF(B)} \)

In \( \mathcal{L}_6^{FF(A)} \) the spin and flavor independently forms scalar

- 25 operators with \( \xi_5 \) pair → 50 operators
  - \( SA, SV, AS, VS, PV, PA, VP, AP, TV, TA, VT, AT, AA, PP, SP, PS, ST, PT, TS, TP, VV, AA, VA, AV, TT \)
  - → No taste-breaking. No derivative terms. Contributes to potential \( \mathcal{V}_6^{FF(A)} \)

In \( \mathcal{L}_6^{FF(B)} \) the spin and flavor are not independent

- 10 operators with \( \xi_5 \) pair → 20 operators
  - \( TV, TA, VT, AT, VV, AA, VA, AV, TT+, TT- \)
  - → Taste-breaking. Derivative terms. No contribution to potential \( \mathcal{V}_6^{FF(B)} \)

*No taste-breaking in ChPT potential up to \( O(a^2) \): SU(2)*