COMPLEX LANGEVIN DYNAMICS AND OTHER APPROACHES AT FINITE CHEMICAL POTENTIAL

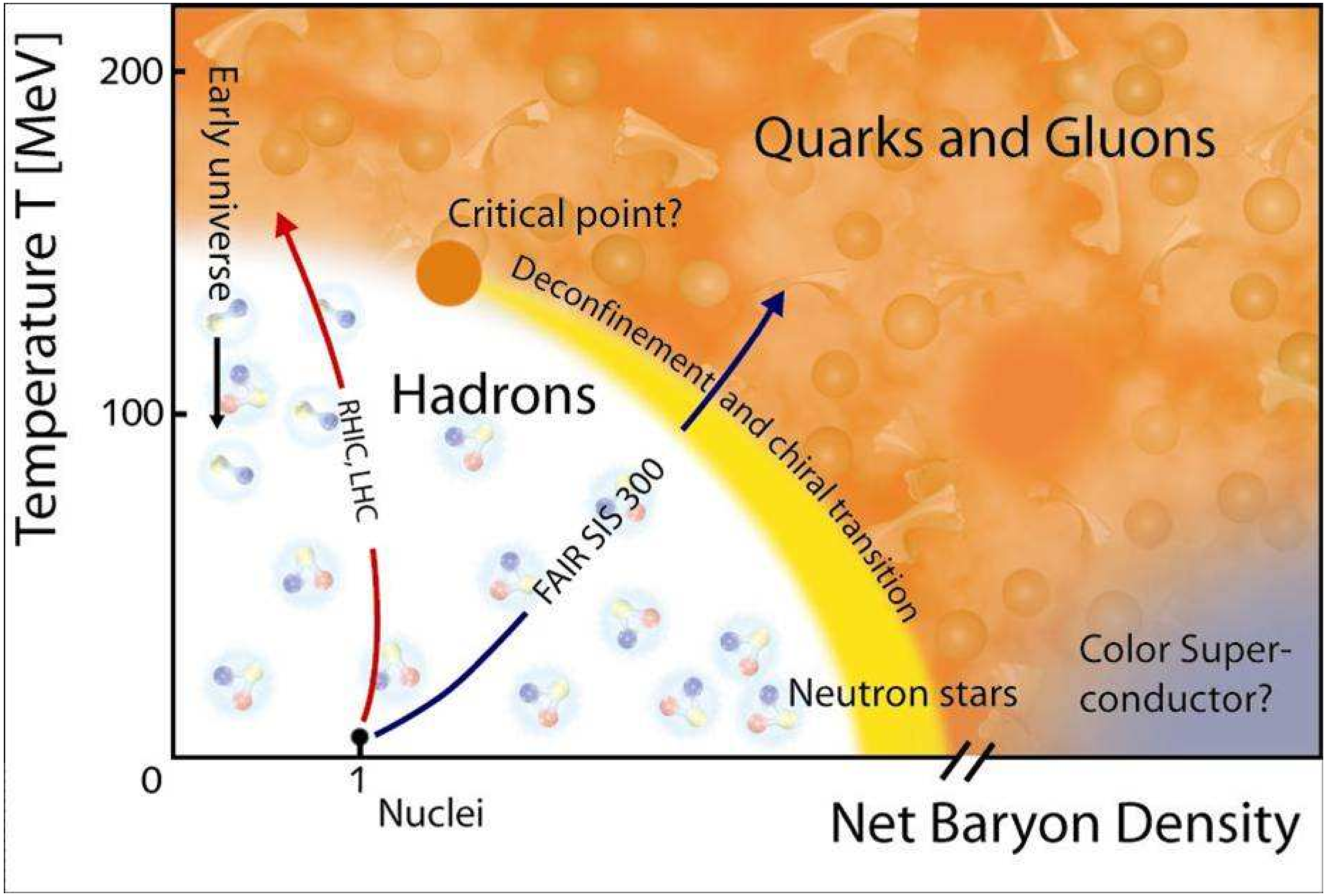
Gert Aarts

Swansea University

Swansea University
Prifysgol Abertawe

QCD PHASE DIAGRAM

A POSSIBILITY



QCD PHASE DIAGRAM

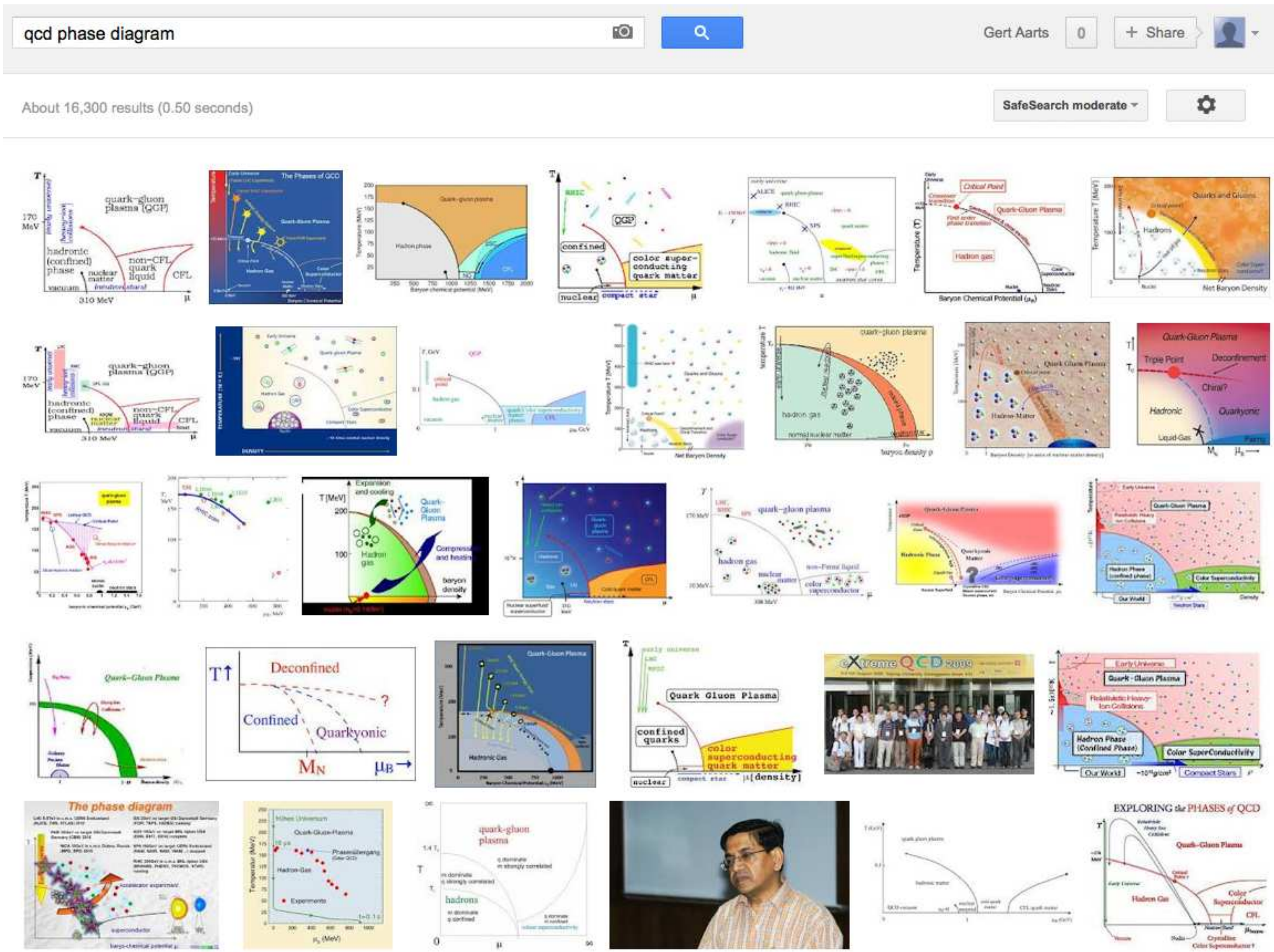
at nonzero quark chemical potential

$$[\det D(\mu)]^* = \det D(-\mu^*)$$

- fermion determinant is complex
- straightforward importance sampling not possible
- sign problem

$\Rightarrow$ phase diagram has not yet been determined non-perturbatively

MANY QCD PHASE DIAGRAMS



OUTLINE

- reminder: sign/overlap/Silver Blaze problems
- techniques that avoid or eliminate sign problem:*
  - sign problem free
  - milder sign problem (3D effective theories)
  - complex Langevin dynamics
- conclusion

* I will not discuss "standard methods" at small μ (reweighting, Taylor series, analytical continuation, histograms) or the strong coupling limit, even though there were many interesting talks at the Conference. I apologize if your work is not discussed!

SIGN/OVERLAP/SILVER BLAZE PROBLEMS

GENERAL REMARKS

integrate out the quarks: complex $\det D(\mu) = |\det D(\mu)|e^{i\theta}$

- sign problem due to complexity, not due to Grassmann nature: also appears in bosonic theories with $\mu \neq 0$
- ignore the phase: $|\det D(\mu)|$, phase quenching (pq)

if pq $\neq$ full, e.g. the pq theory has a transition to a high-density phase at a lower critical μ than the full theory:

- bad overlap problem: average sign

$$\langle e^{i\theta}\rangle_{\rm pq} = Z/Z_{\rm pq} = e^{-\Omega\Delta f} \qquad\qquad \Delta f = f - f_{\rm pq}$$

  vanishes exponentially with 4-volume Ω
- Silver Blaze problem: many cancelations to ensure that onset happens at the right critical μ Cohen 03

SIGN/OVERLAP/SILVER BLAZE PROBLEMS

GENERAL REMARKS

example: $N_f = 2$ QCD with $[\det D(\mu)]^2$

- phase-quenched: $|\det D(\mu)|^2 = \det D(\mu) \det D(-\mu)$
  $\Rightarrow$ isospin chemical potential

at $T = 0$:

- isospin: onset at $\mu = m_\pi/2$
  full: onset at $\mu \sim m_N/3$ (– binding energy)
- Silver Blaze region: $m_\pi/2 < \mu \lesssim m_N/3$
- intricate cancelations, e.g. eigenvalue density of Dirac operator is complex, highly oscillatory, with exp. large amplitude in thermodynamic limit
- precise integration to get correct cancelations

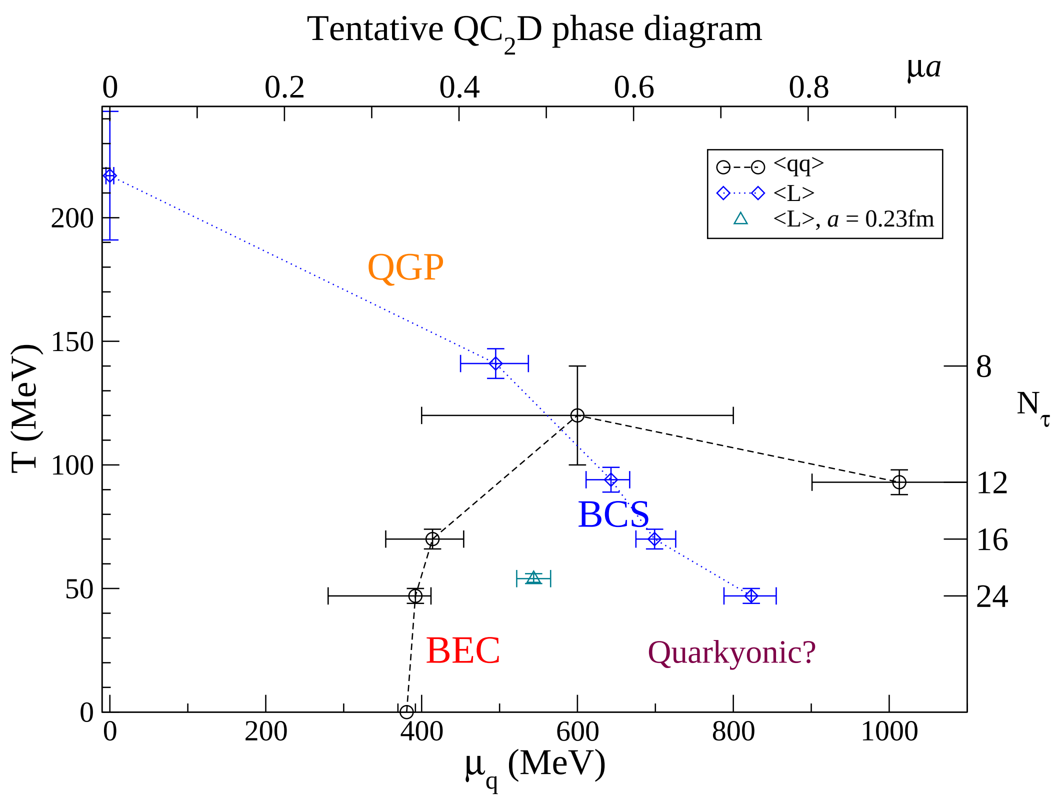
Osborn, Splittorff & Verbaarschot 05

THEORIES WITHOUT A SIGN PROBLEM

DETERMINANT IS REAL, EVEN WHEN $\mu > 0$

prime example: two-color QCD (QC$_2$D)

phase diagram can be determined Skullerud et al 12



diquark condensate

correct in QC$_2$D

detailed analysis of singular values $D^\dagger(\mu)D(\mu)\psi_n = \xi_n^2\psi_n$

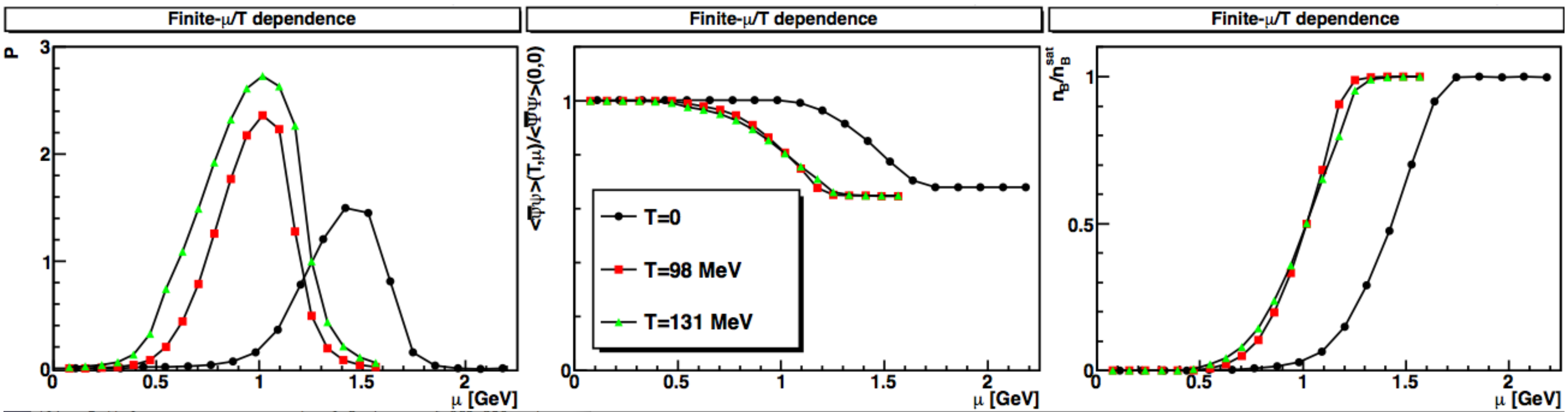
(independent of Dirac eigenvalues) Wettig et al 12

THEORIES WITHOUT A SIGN PROBLEM

DETERMINANT IS REAL, EVEN WHEN $\mu > 0$

recent example: exceptional gauge group G$_2$

bosonic (qq) *and* fermionic baryons (qqq) $+$ hybrids (qggg)



- onset presumably at diquark threshold, but indications for a second rise in density at larger μ (not shown)
- requires more precise spectrum calculations
- avoid saturation

Maas, Wellegehausen et al 12

THEORIES WITH MILDER SIGN PROBLEM

STRATEGY

effective field theories/spin models may

- have a milder sign problem
- be reformulated without sign problem (flux, world line)*
- be solvable with complex Langevin

ideally

- detailed mapping between effective couplings and QCD parameters
- use insight for QCD phase diagram

* see also Chandrasekharan & Li for fermion models

EFFECTIVE SPIN MODELS

SIMPLEST EXAMPLE

3-dimensional SU(3) spin model

$$S = S_B + S_F$$

$$S_B = -\beta \sum_{<xy>} \left[P_x P_y^* + P_x^* P_y \right]$$

$$S_F = -h \sum_{x} \left[e^{\mu} P_x + e^{-\mu} P_x^* \right]$$

- SU(3) matrices: $P_x = \mathrm{Tr}\, U_x$
- gauge action: nearest neighbour Polyakov loops
- (static) quarks represented by Polyakov loops
- complex action $S_F^*(\mu) = S_F(-\mu^*)$

EFFECTIVE SPIN MODELS

SIMPLEST EXAMPLE

very, very, very old model:

- mean field solutions — Banks & Ukawa 83
- complex Langevin — Karsch & Wyld 85

  Bilic, Gausterer & Sanielevici 88

recent revival:

- reformulated as a flux model without sign problem

  Gattringer 11 (& Mercado 12)
- re-addressed with complex Langevin — GA & James 11
- mean field analysis — Greensite & Splittorff 12
- part of family of high-order strong-coupling models

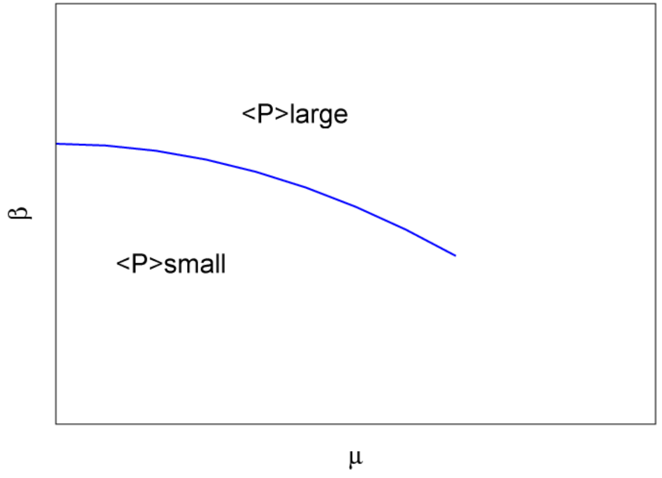
  Philipsen et al 11

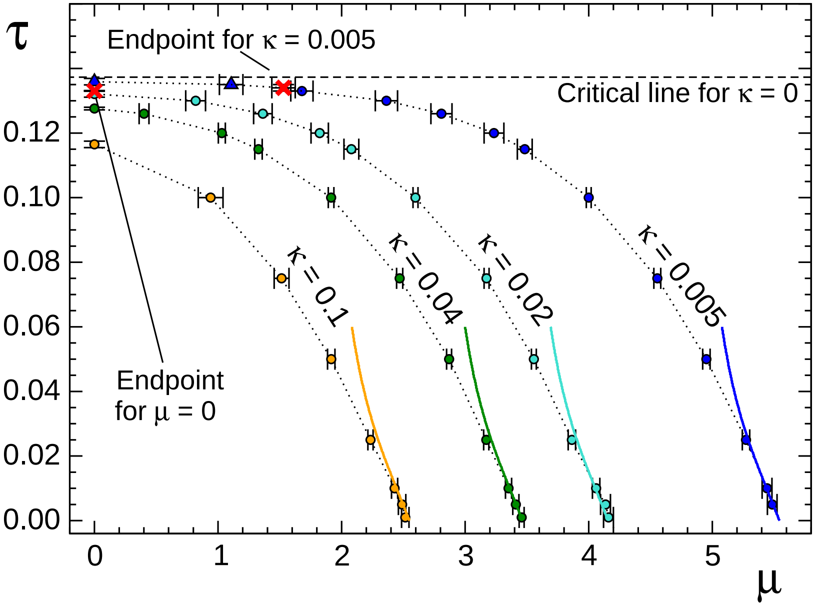
SU(3) SPIN MODEL

GATTRINGER & MERCADO 12

flux representation (“high temperature” expansion):

- expand Boltzmann weight
- integral over powers of Polyakov loops at each site
- find a monomer-dimer system with constraints
- no sign problem: solved with Monte Carlo (or worm)





EFFECTIVE SPIN MODELS

SYSTEMATIC EXTENSION

construct whole family of effective Polyakov loop models

Philipsen, Langelage, Lottini & Fromm 10, 11

- integrate out spatial links in strong-coupling expansion
- systematically improvable
- include more Polyakov loop interactions

schematic form of action (without fermions):

$$S = \lambda_1 \sum_{<xy>} \left(P_x P_y^* + P_x^* P_y\right) + \lambda_2 \sum_{[xy]} \left(P_x P_y^* + P_x^* P_y\right)$$

$$+(\text{higher order representations}) + \ldots$$

EFFECTIVE SPIN MODELS

SYSTEMATIC EXTENSION

resum certain subclasess to all orders:

$$S = \lambda_1 \sum_{<xy>} \left(P_x P_y^* + c.c.\right) + \cdots = \sum_{<xy>} \ln\left[1 + \lambda_1 (P_x P_y^* + c.c.)\right] + \ldots$$

effective couplings λ_i depend on β and N_τ, e.g.:

$$\lambda_1(u, N_\tau = 4) = u^4 \exp\left[4\left(4u^4 + 12u^5 - 14u^6 + \ldots + \frac{1035317}{5120}u^{10} + \ldots\right)\right]$$

$$\lambda_1(u, N_\tau \geq 5) = u^{N_\tau} \exp\left[N_\tau\left(4u^4 + 12u^5 - 14u^6 + \ldots + \frac{1055797}{5120}u^{10} + \ldots\right)\right]$$

$$\lambda_2(u, N_\tau = 4) = u^8 \left[12u^2 + 26u^4 + 364u^6 + \ldots\right]$$

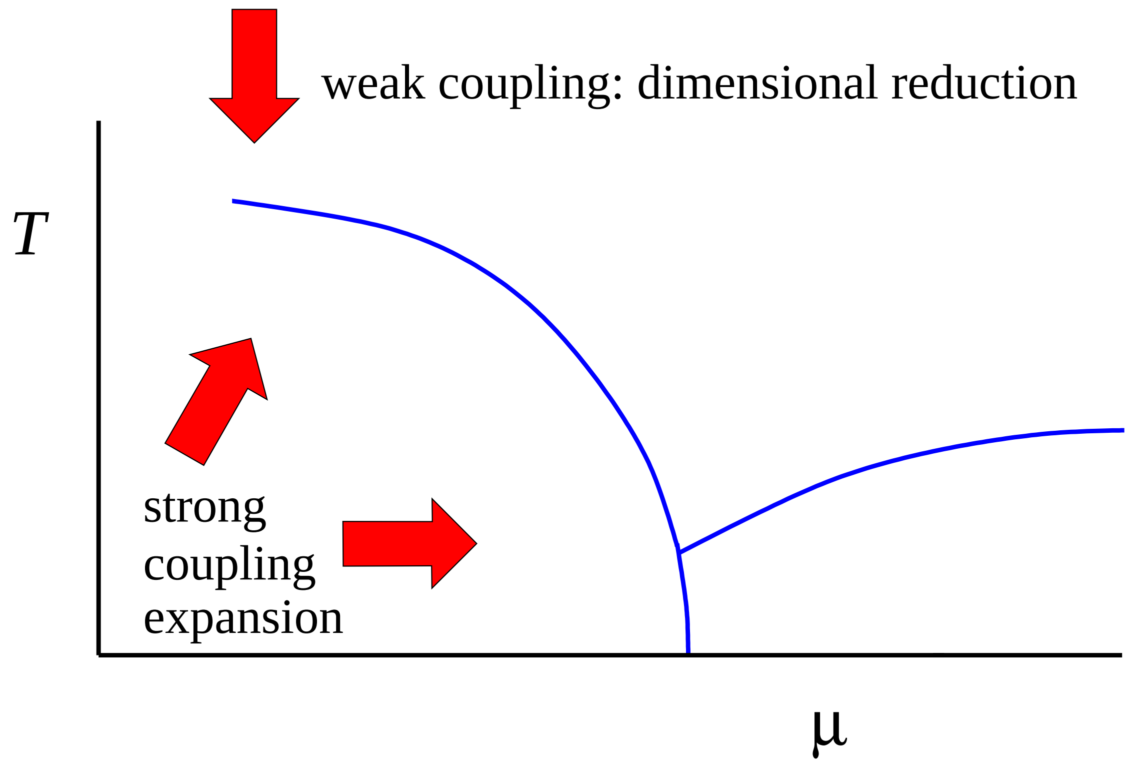
$u = u(\beta) = a_f(\beta) = \beta/18 + \ldots$ (character expansion)

$\Rightarrow$ far beyond simplest SU(3) spin model
quantitative predictions?

EFFECTIVE SPIN MODELS

SYSTEMATIC EXTENSION

- forget strong coupling origin: view as effective model
- complementary to dimensional reduction at high T and weak coupling
- test range of validity

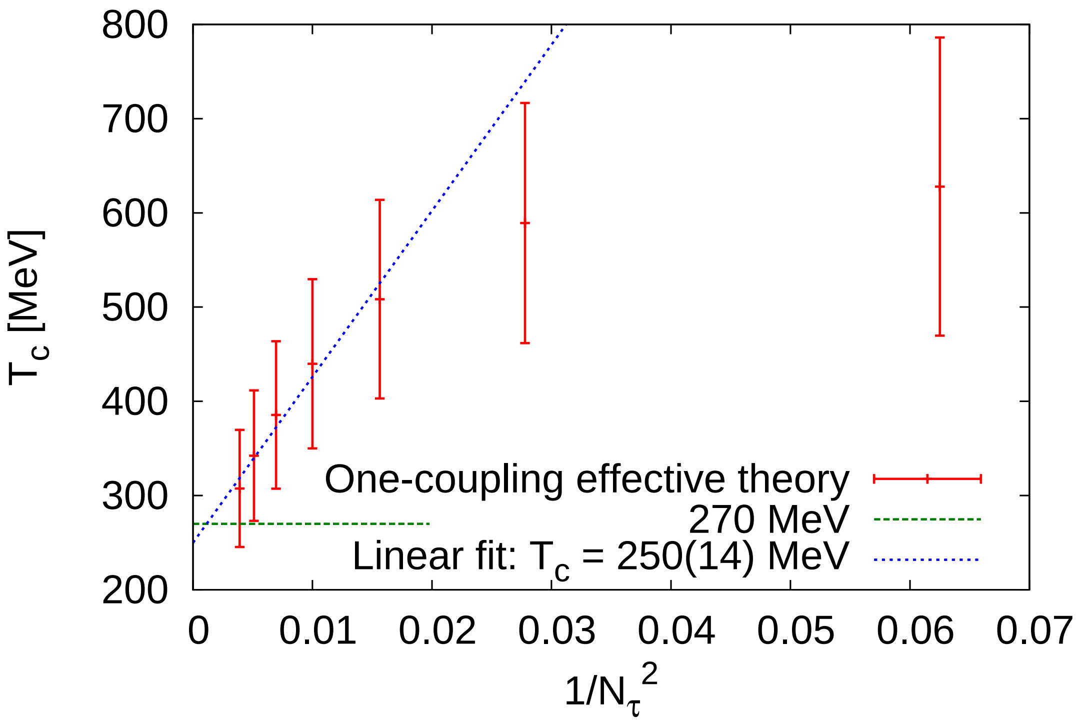


EFFECTIVE SPIN MODELS

SYSTEMATIC EXTENSION

example: pure gauge SU(3) Philipsen, Langelage & Lottini 10

- determine one critical coupling $\lambda_{1,c}$ in 3D
- relate $\lambda_{1,c}(\beta, N_\tau)$ to 4D critical coupling $\beta_c(N_\tau)$



- reproduce 4D results up to $N_\tau = 16$

Spin models with quarks

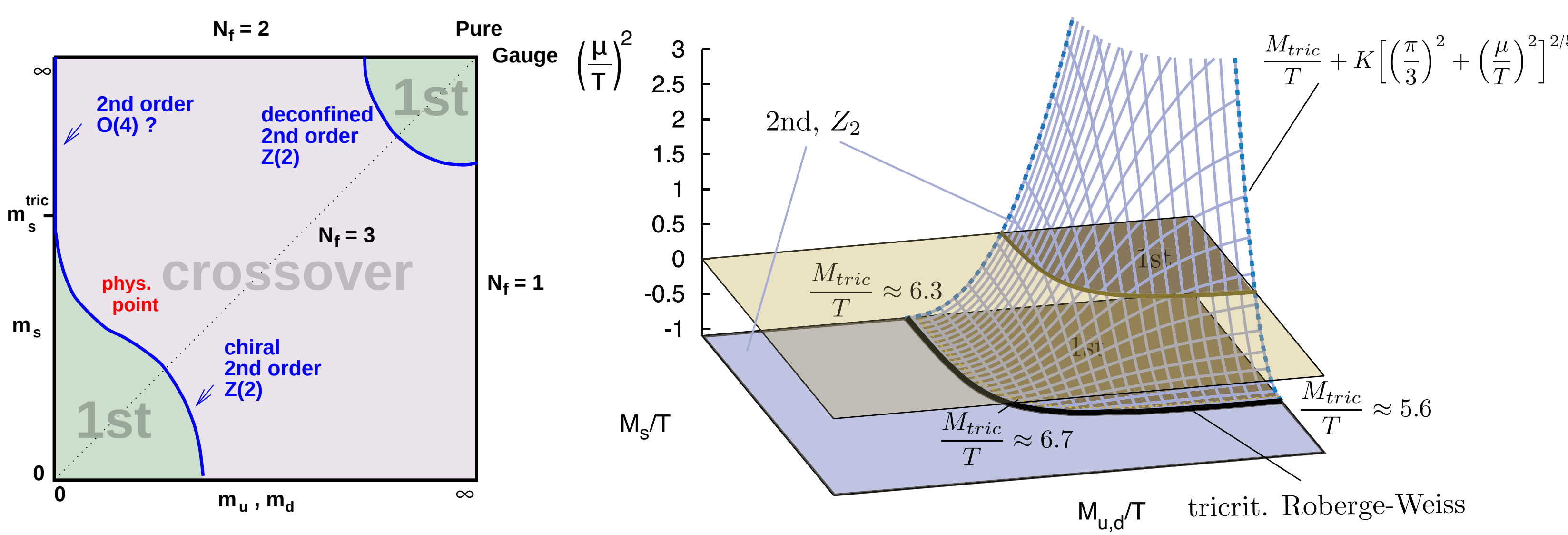
HEAVY QUARKS AND NONZERO CHEMICAL POTENTIAL

- hopping expansion

$$S_F \sim -\ln\det\left[(1+he^{\mu/T}W_x)(1+he^{-\mu/T}W_x^\dagger)\right]$$

$$W_x = \text{untraced Polyakov loop}, h = h(\kappa, N_\tau)$$

- determine phase diagram in heavy quark sector
- mild sign problem (reweighting)



SPIN MODELS WITH QUARKS

HEAVY QUARKS AND NONZERO CHEMICAL POTENTIAL

rich phase structure: interplay with centre-symmetry

RW line at $\mu_{\rm RW} = i\pi T/3$, RW endpoint at $T_{\rm RW}$:

- first order (triple) point for light and heavy quarks
- second order endpoint for intermediate mass quarks

Roberge-Weiss 86, d'Elia-Sanfilippo 09, de Forcrand-Philipsen 10

$\Rightarrow$ tricritical line in Columbia plot at $(\mu/T)^2 = -(\pi/3)^2$

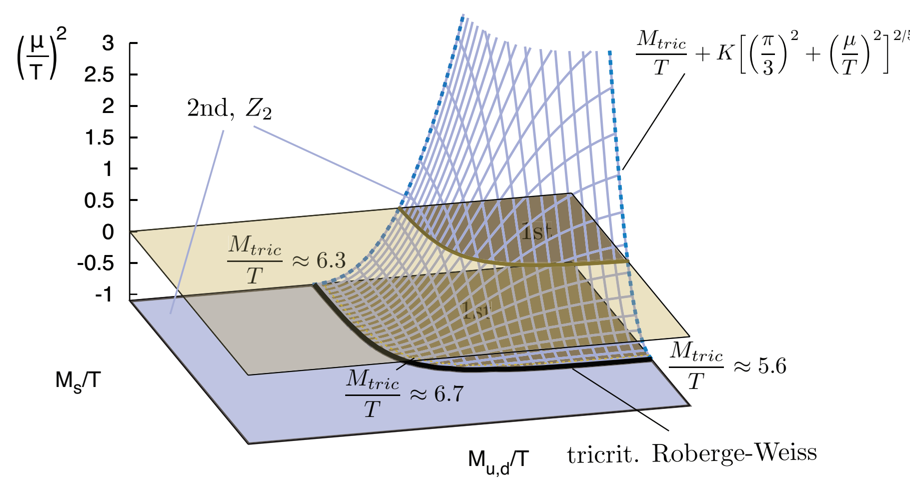
tricritical scaling determines curvature of critical surface

de Forcrand & Philipsen 10

Fromm-Langelage-Lottini-Philipsen 11

also in 3-state Potts model

Kim, de Forcrand et al 06



COMPLEX INTEGRALS

- consider simple integral

$$Z(a,b) = \int_{-\infty}^{\infty} dx\, e^{-S(x)} \qquad\qquad S(x) = ax^2 + ibx$$

- complete the square/saddle point approximation:

  into complex plane

- lesson: don't be real(istic), be more imaginative

radically different approach:

- complexify all degrees of freedom $x \to z = x + iy$
- enlarged complexified space
- new directions to explore

COMPLEX SADDLE POINTS

recent proposal (as an alternative to complex Langevin):

high-density QCD on a Lefschetz thimble?

Cristoforetti, di Renzo & Scorzato 12

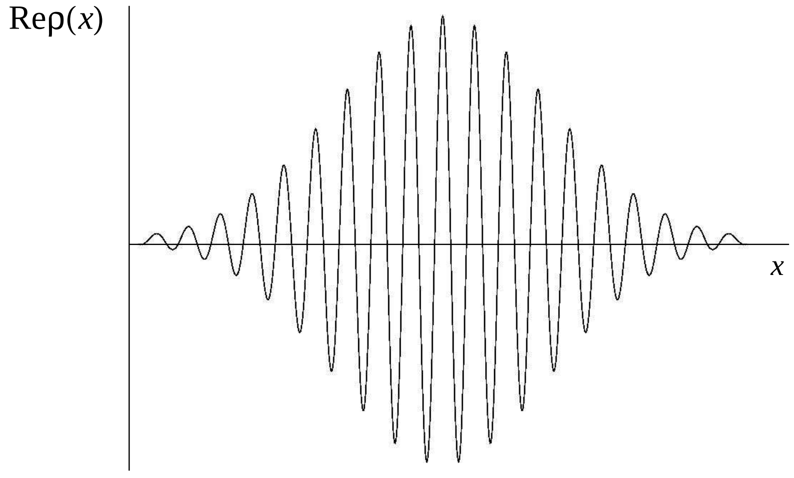
- deform integration contour into complex plane
- associate (real) integration domain with each stationary point (“thimble”)
- path integral $=$ sum over integrals over thimbles
- imaginary parts should be subleading
- motivated by Witten and Morse theory (!)

interesting idea – wait for implementation

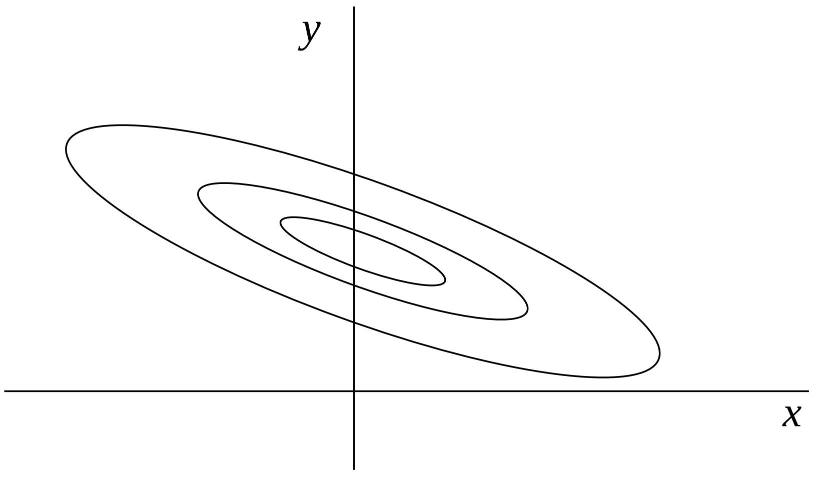
COMPLEXIFIED FIELD SPACE

COMPLEX WEIGHT $\rho(x)$

dominant configurations in the path integral?



$\Rightarrow$



real and positive distribution $P(x, y)$: how to obtain it?

$\Rightarrow$ solution of stochastic process

complex Langevin dynamics

Parisi 83, Klauder 83

COMPLEX LANGEVIN DYNAMICS

COMPLEX WEIGHT $\rho(x)$

does it work?

- for real actions: stochastic quantization
- equivalent to path integral quantization

Damgaard & Hüffel, Phys Rep 87

- for complex actions: no formal proof
- troubled past: “disasters of various degrees”

Ambjørn et al 86

why talk about it here? recent examples in which CL

- can solve Silver Blaze problem
- can handle severe sign problems
- gives the correct result (!)
- improved analytical understanding

COMPLEX LANGEVIN DYNAMICS

various scattered results since mid 1980s

here:

review results obtained mostly with IO Stamatescu, E Seiler and F James, starting with hep-lat/0807.1597

also with D Sexty, with K Splittorff, with J Pawlowski

- Silver Blaze problem
- severe sign problem
- analytical insight
- SU(3) spin model
- more analytical insight

SILVER BLAZE AND SIGN PROBLEM

BOSE GAS

relativistic Bose gas at nonzero μ, $d = 4$, $m^2 > 0$

- continuum action

$$S = \int d^4x \left[|\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu \left(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi\right) + \lambda |\phi|^4 \right]$$

linear term in μ is imaginary

- lattice action

$$S = \sum_x \left[\left(2d + m^2\right) \phi_x^* \phi_x + \lambda \left(\phi_x^* \phi_x\right)^2 - \sum_{\nu=1}^{4} \left(\phi_x^* e^{-\mu \delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^* e^{\mu \delta_{\nu,4}} \phi_x \right) \right]$$

- $S^*(\mu) = S(-\mu^*)$ as always

GA 08, 09

SILVER BLAZE AND SIGN PROBLEM

BOSE GAS

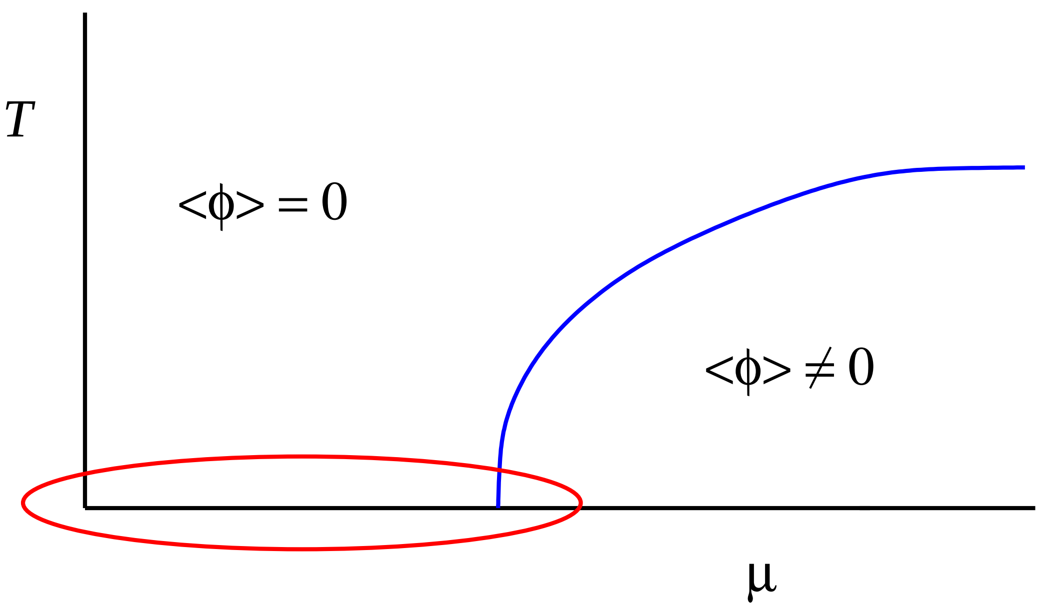
at $T = 0$:

- full theory: μ independence until onset, $\mu_c \sim m$
- phase-quenched theory: always μ dependence

$$V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

phase diagram
in full theory

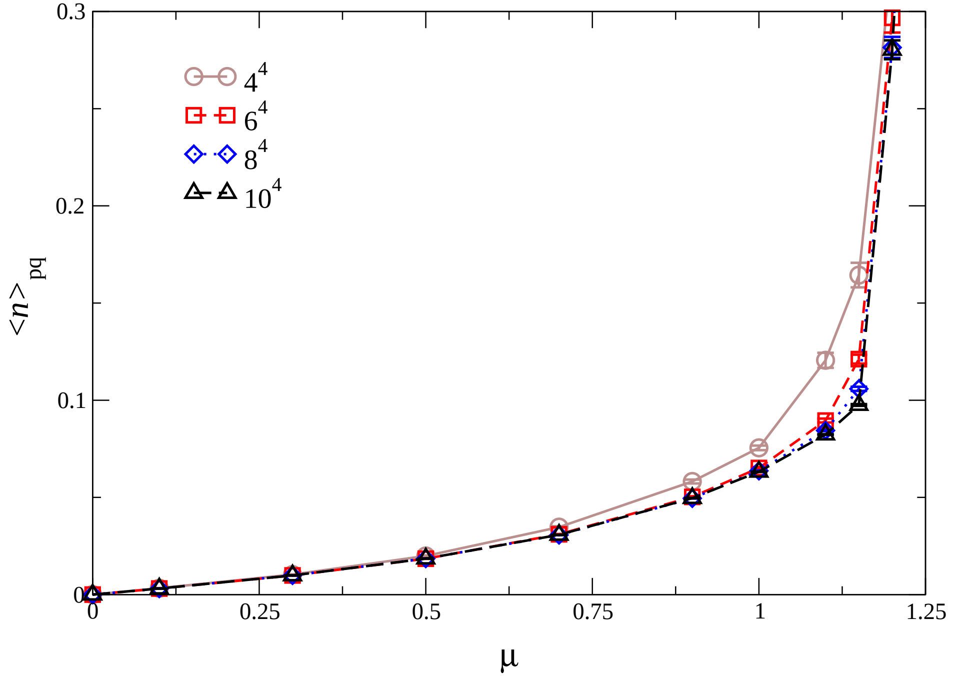
Silver Blaze
problem

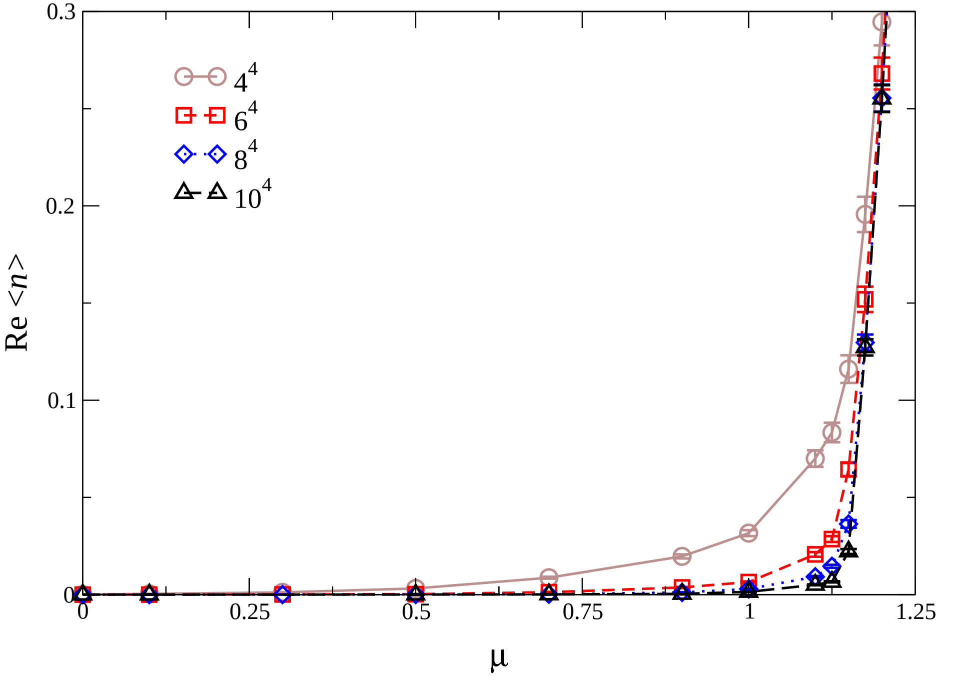


SIGN AND SILVER BLAZE PROBLEMS

BOSE GAS

density: $(m = \lambda = 1)$





phase quenched full

Silver Blaze problem resolved

phase $e^{i\theta} = e^{-S}/|e^{-S}|$ does precisely what is expected

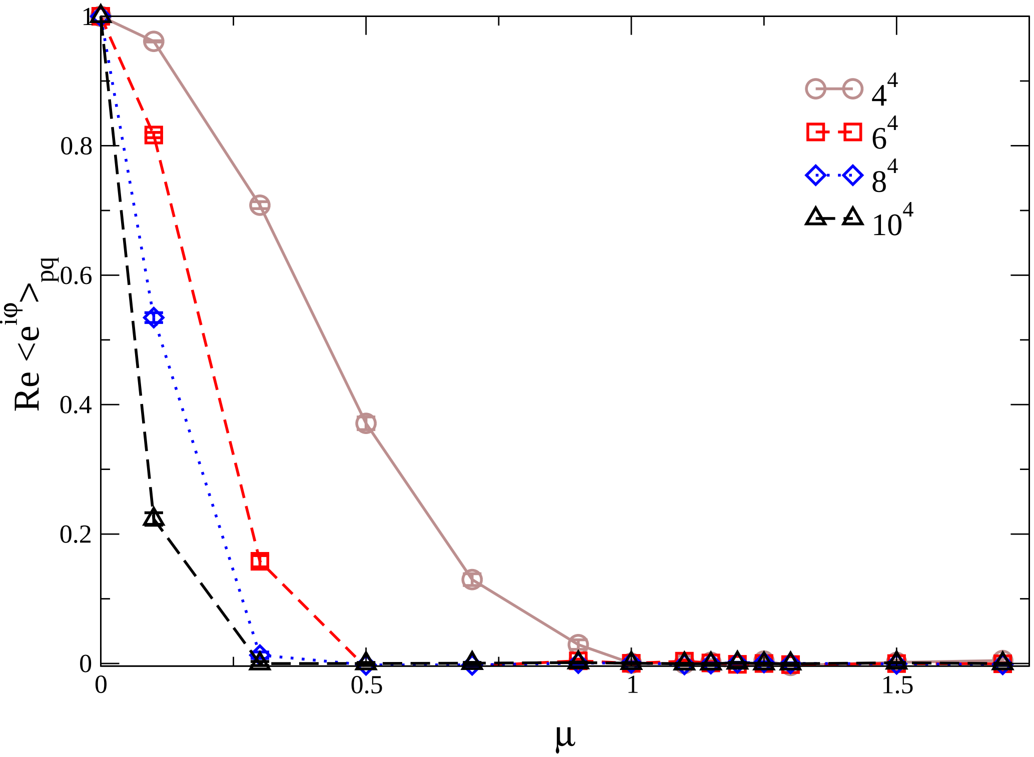
HOW SEVERE IS THE SIGN PROBLEM?

AVERAGE PHASE FACTOR

- complex action $e^{-S} = |e^{-S}|e^{i\theta}$
- average phase factor in phase quenched theory

$$\langle e^{i\theta}\rangle_{\rm pq} = \frac{Z}{Z_{\rm pq}} = e^{-\Omega\Delta f}$$

$$\Omega \to \infty, \mu \neq 0$$



- exponentially hard in thermodynamic limit

ONE-DIMENSIONAL QCD

SILVER BLAZE

- exactly solvable — Gibbs 86, Bilic & Demeterfi 88
- phase quenched: transition at $\mu = \mu_c$, full: no transition

severe sign problem when $|\mu| > |\mu_c|$

- chiral condensate:
  write as integral over spectral density

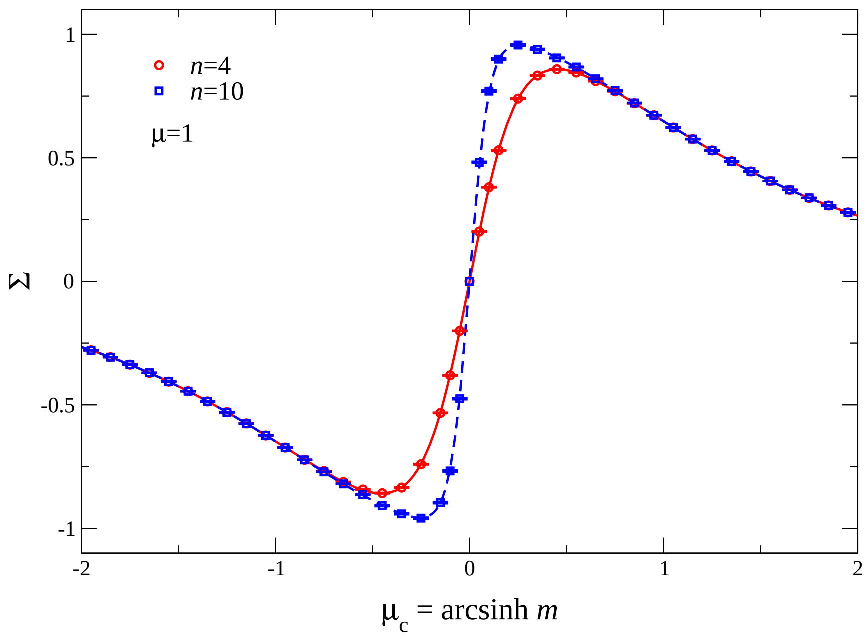
$$\Sigma = \int d^2 z \, \frac{\rho(z;\mu)}{z+m} \qquad\qquad \mu_c = \text{arcsinh}\, m$$

- $\rho(z;\mu)$ complex and oscillatory — Ravagli & Verbaarschot 07
- condensate independent of μ: Silver Blaze
- solve with complex Langevin — GA & Splittorff 10

ONE-DIMENSIONAL QCD

SILVER BLAZE

- exact results reproduced
- discontinuity at $\mu_c = 0$ in thermodynamic limit $n \to \infty$



- sign problem severe when $|\mu_c| < |\mu|$
- condensate independent of μ: Silver Blaze

ONE-DIMENSIONAL QCD

ANALYTIC SOLUTION IN THERMODYNAMIC LIMIT

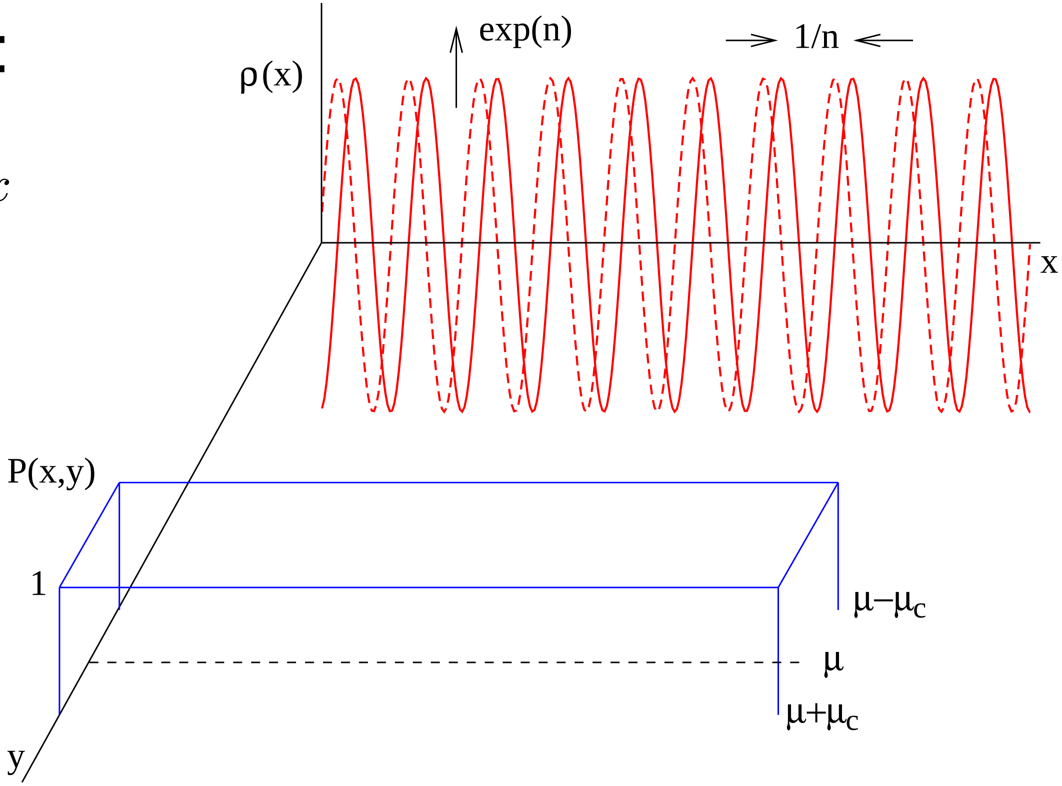
elegant analytical solution:

- original distribution:

$$\rho(x) \sim e^{n(\mu-\mu_c)} e^{inx}$$

when $n \to \infty$

- real distribution sampled by complex Langevin:



$$P(x,y) = \begin{cases} 1 & \mu - \mu_c < y < \mu + \mu_c \\ 0 & \text{elsewhere} \end{cases}$$

TROUBLED PAST

OF COMPLEX LANGEVIN DYNAMICS

1. numerical problems: runaways, instabilities

   $\Rightarrow$ adaptive stepsize

   no instabilities observed, works for SU(3) gauge theory

   `GA, FJ, ES & IOS 09`

   `a la Ambjorn et al 86`

2. theoretical status unclear

   $\Rightarrow$ detailed analyis, identified necessary conditions

   `GA, FJ, ES & IOS 09-12`

3. convergence to wrong limit

   $\Rightarrow$ better understood but not yet resolved

   `in progress`

ANALYTICAL UNDERSTANDING

SKETCH

consider expectation values and Fokker-Planck equations

one degree of freedom x, complex action $S(x)$, $\rho(x) \sim e^{-S(x)}$

- wanted: $\langle O(x,t)\rangle_\rho = \int dx\ \rho(x,t)O(x)$

$$\partial_t \rho(x,t) = \partial_x \left(\partial_x + S'(x)\right) \rho(x,t)$$

- solved with CLE:

$$\langle O(x,t)\rangle_P = \int dxdy\ P(x,y;t)O(x+iy)$$

$$\partial_t P(x,y;t) = \left[\partial_x \left(\partial_x - K_x\right) - \partial_y K_y\right] P(x,y;t)$$

with $K_x = -\mathrm{Re}S'$, $K_y = -\mathrm{Im}S'$

- question: $\langle O(x,t)\rangle_P = \langle O(x,t)\rangle_\rho$?

ANALYTICAL UNDERSTANDING

SKETCH

question: $\langle O(x,t)\rangle_P = \langle O(x,t)\rangle_\rho$ as $t \to \infty$?

answer: yes, provided some conditions are met:

- distribution $P(x,y)$ should drop off fast enough in y direction
- partial integration without boundary terms possible
- actually $O(x+iy)P(x,y)$ for large enough set $O(x)$

$\Rightarrow$ distribution should be sufficiently localized

- can be tested numerically via criteria for correctness

$$\langle LO(x+iy)\rangle = 0$$

with L Langevin operator 0912.3360, 1101.3270

SU(3) SPIN MODEL

DETAILED TESTS

apply these ideas to 3D SU(3) spin model GA & James 11

- earlier solved with complex Langevin Karsch & Wyld 85
  Bilic, Gausterer & Sanielevici 88
- however, no detailed tests performed

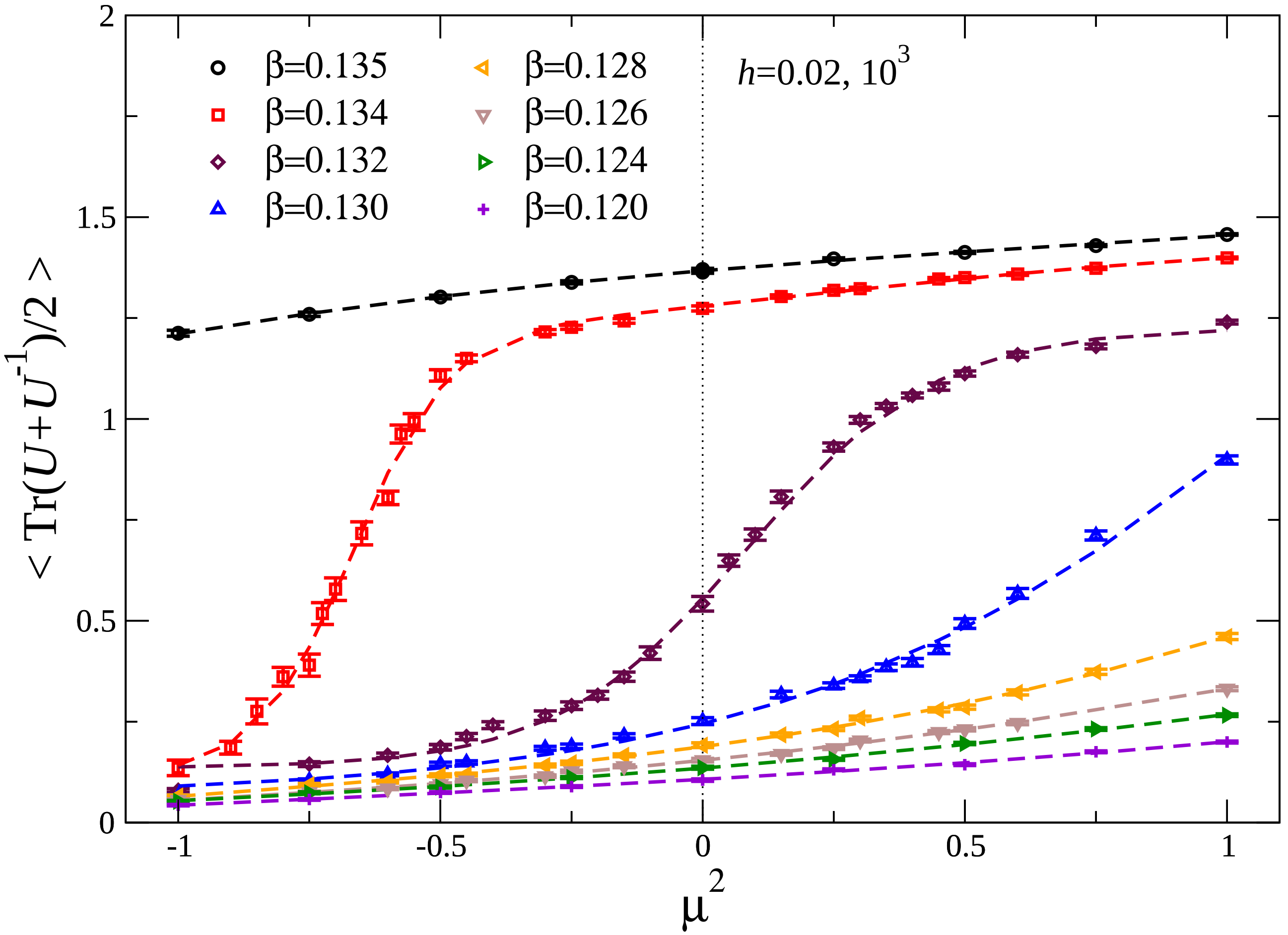
$\Rightarrow$ test reliability of complex Langevin using developed tools

- analyticity in μ^2:
  - from imaginary to real μ
  - Taylor series
- criteria for correctness
- comparison with flux formulation Gattringer & Mercado 12

SU(3) SPIN MODEL

REAL AND IMAGINARY POTENTIAL

first-order transition in $\beta - \mu^2$ plane, $\langle P + P^* \rangle / 2$

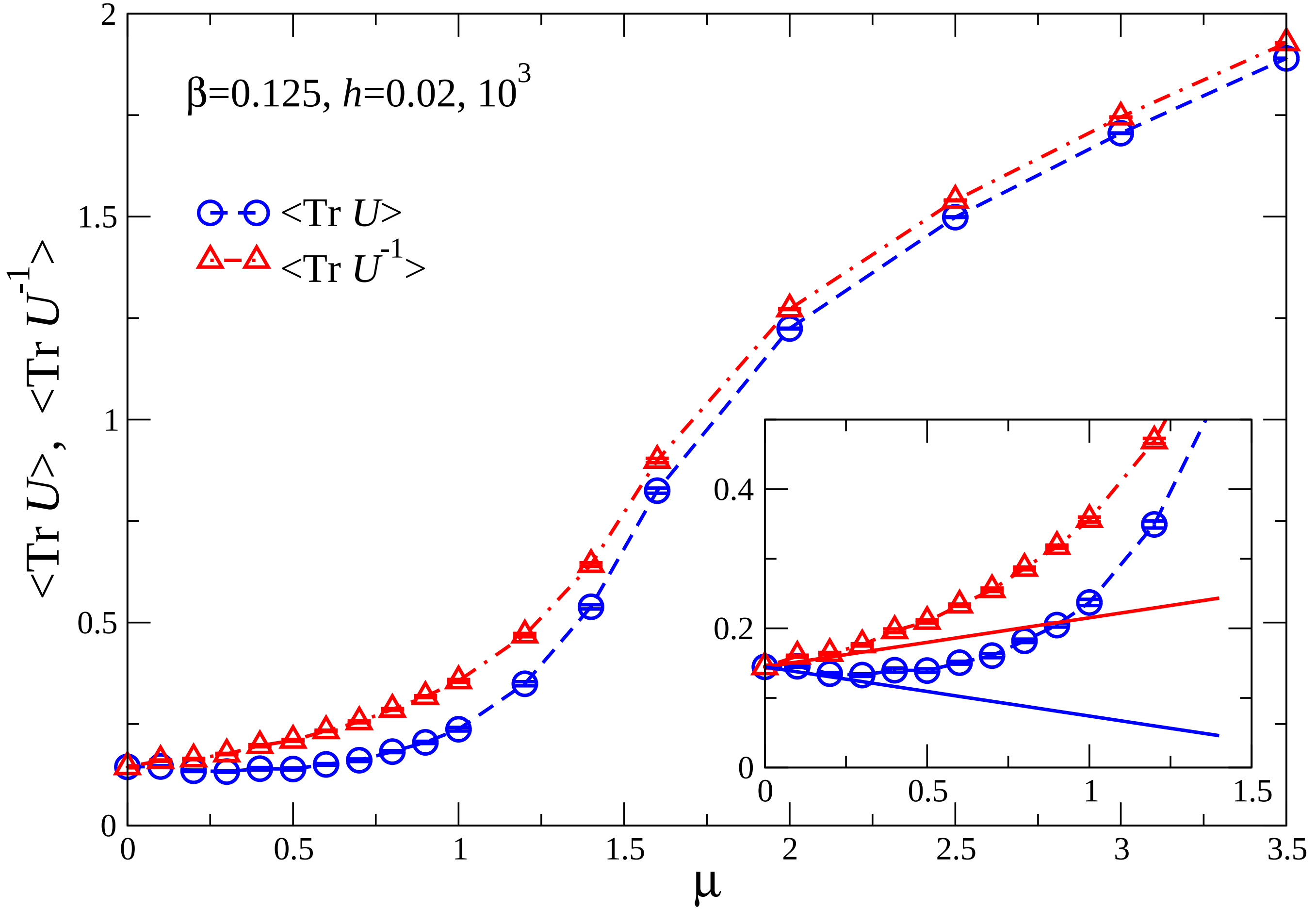


negative μ^2: real Langevin — positive μ^2: complex Langevin

SU(3) SPIN MODEL

REAL CHEMICAL POTENTIAL

immediate splitting between $\langle P \rangle$ and $\langle P^* \rangle$: no Silver Blaze



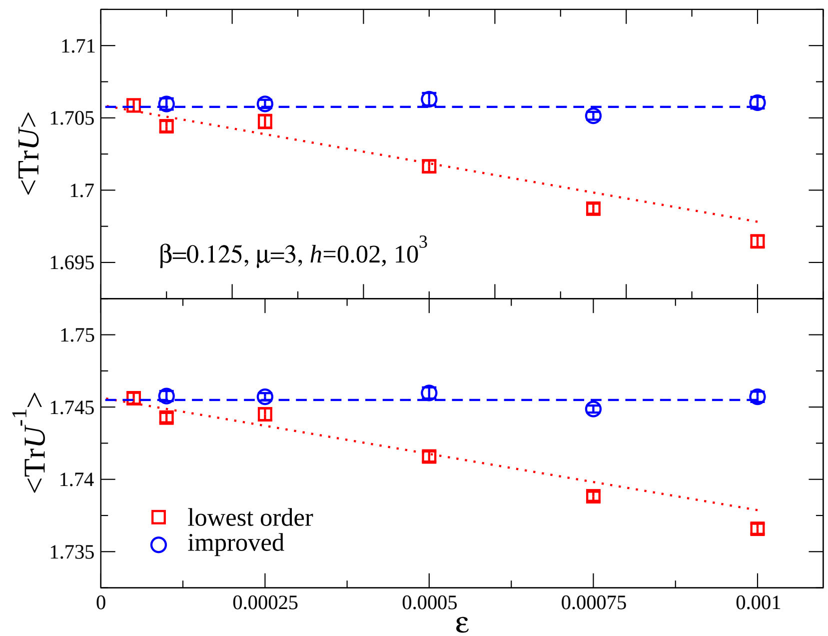
inset: lines from first-order Taylor expansion

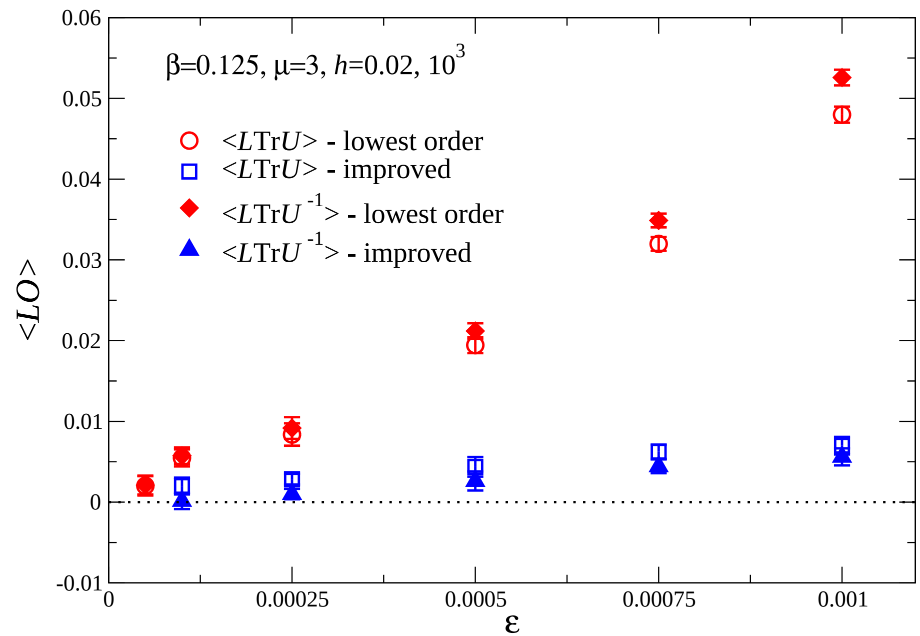
SU(3) SPIN MODEL

STEPSIZE DEPENDENCE

left: $\langle P \rangle$ (top) and $\langle P^* \rangle$ (bottom) at $\mu = 3$

right: criteria for correctness $\langle LO \rangle = 0$





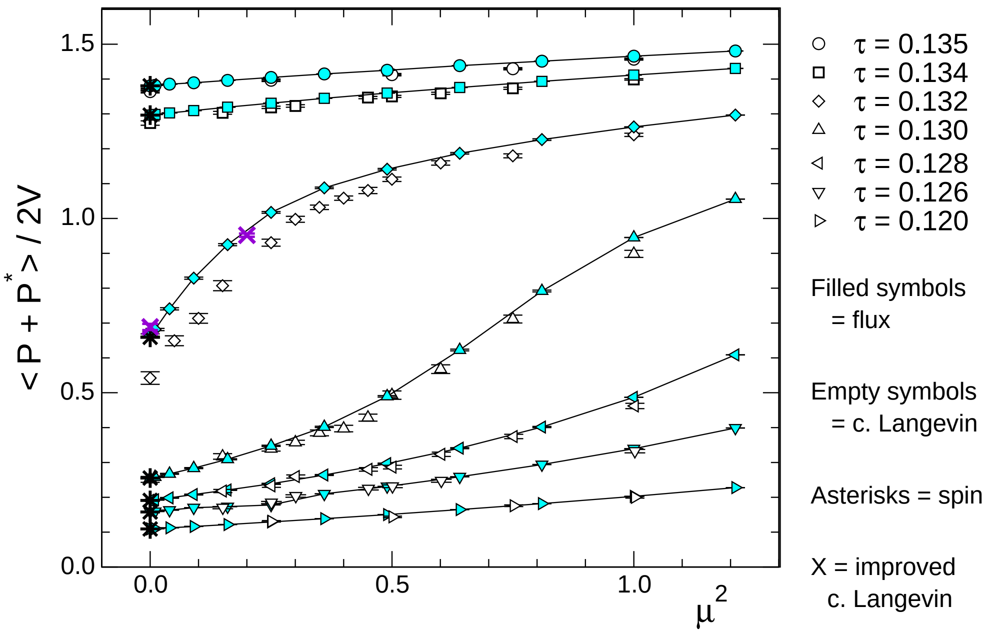
improved stepsize algorithm to eliminate linear dependence

criteria satisfied as stepsize $\epsilon \to 0$

SU(3) SPIN MODEL

FLUX REPRESENTATION

comparison with result obtained using flux representation



CL: finite stepsize errors in lowest-order algorithm

improved algorithm removes discrepancy in critical region

complex Langevin passes all the tests: why?

- localized distribution: fast decay in imaginary direction
- real manifold is stable under small fluctuations
- Haar measure plays essential role

STABILIZING DRIFT

HAAR MEASURE, JACOBIANS

- Haar measure contribution to complex drift restoring
- controlled exploration of the complex field space
- compare with XY/U(1) model: trivial Haar measure
  CLE fails in part of phase diagram GA & James 10

employ this: generate Jacobian by field redefinition

$$Z = \int dx\, e^{-S(x)} \qquad x = x(u) \qquad J(u) = \frac{\partial x(u)}{\partial u}$$

$$= \int du\, e^{-S_{\text{eff}}(u)} \qquad S_{\text{eff}}(u) = S(u) - \ln J(u)$$

drift: $K(u) = -S'_{\text{eff}}(u) = -S'(u) + J'(u)/J(u)$

which field redefinition? singular at $J(u) = 0$ but should be restoring in complex plane

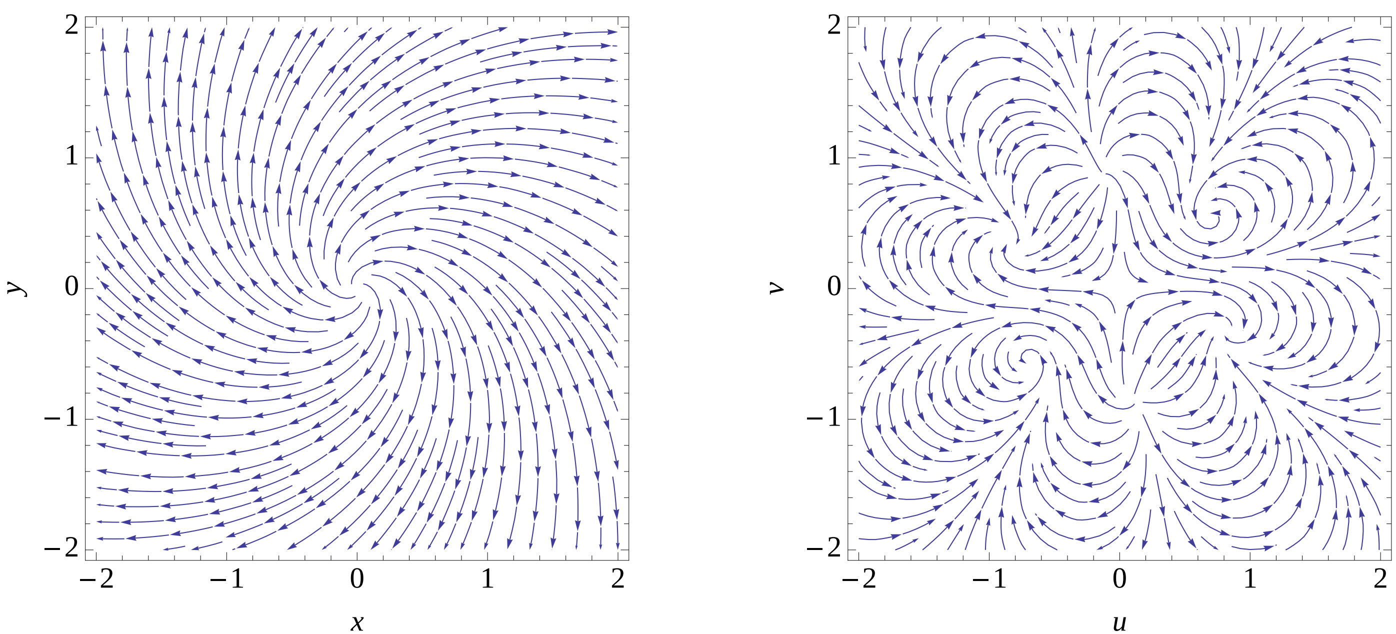
FUN WITH COMPLEX LANGEVIN

STABILIZING JACOBIANS

Gaussian example: defined when $\mathrm{Re}(\sigma) = a > 0$

$$Z = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\sigma x^2} \qquad \sigma = a + ib \qquad \langle x^2 \rangle = \frac{1}{\sigma}$$

what if $a < 0$? flow in complex space for $a = -1, b = 1$:



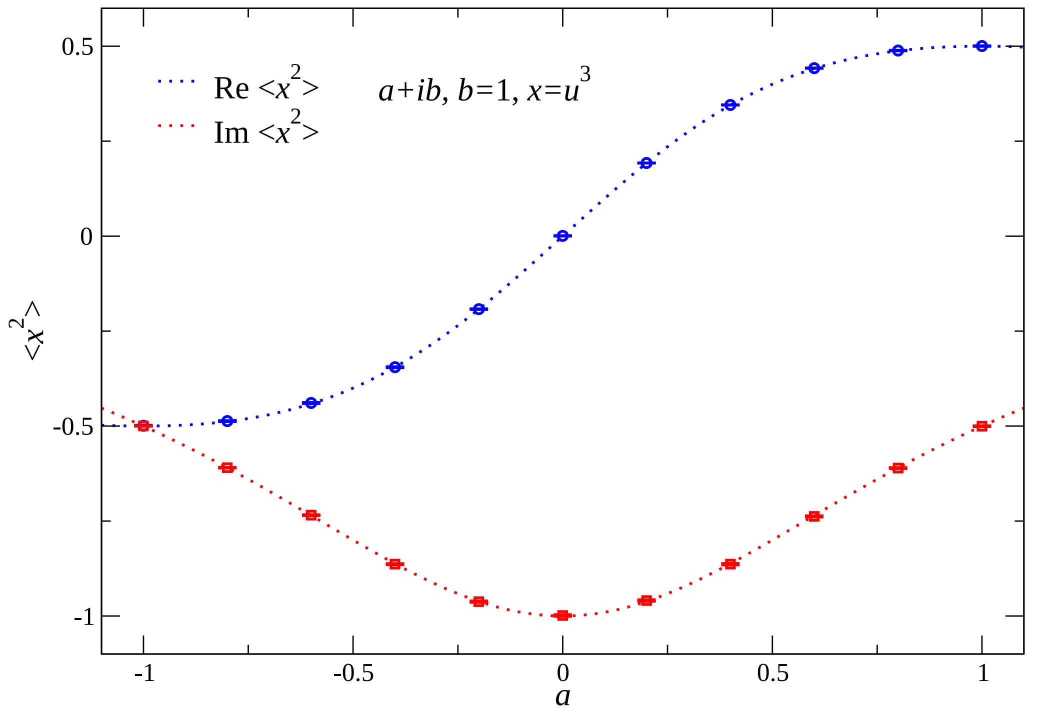
left: highly unstable

right: after transformation $x(u) = u^3$
attractive fixed points

FUN WITH COMPLEX LANGEVIN

STABILIZING JACOBIANS

do CLE in the u formulation and compute $\langle x^2 \rangle = \langle u^6 \rangle$



$$\langle x^2 \rangle = \frac{1}{\sigma} = \frac{a - ib}{a^2 + b^2}$$

take also negative a

CLE finds the analytically continued answer to negative a!

clearly needs more exploration – potential for stabilization – affects convergence

SUMMARY AND OUTLOOK

prediction for the heavy quark corner

- will be settled, using a combination of techniques and effective models

challenge: light quarks

- strong sensitivity to quark masses
- effective models much less predictive

complex Langevin dynamics can handle

- sign problem
- Silver Blaze problem
- phase transition
- thermodynamic limit

in a variety of theories, but . . .

SUMMARY AND OUTLOOK

- . . . correct result not guaranteed
- . . . can convergence to wrong result

we have

- developed better understanding
- formulated criteria for correctness
- a theoretical framework also applicable to SU(3) lattice theory

tests of various ideas in SU(3) in progress

not yet exhausted . . .